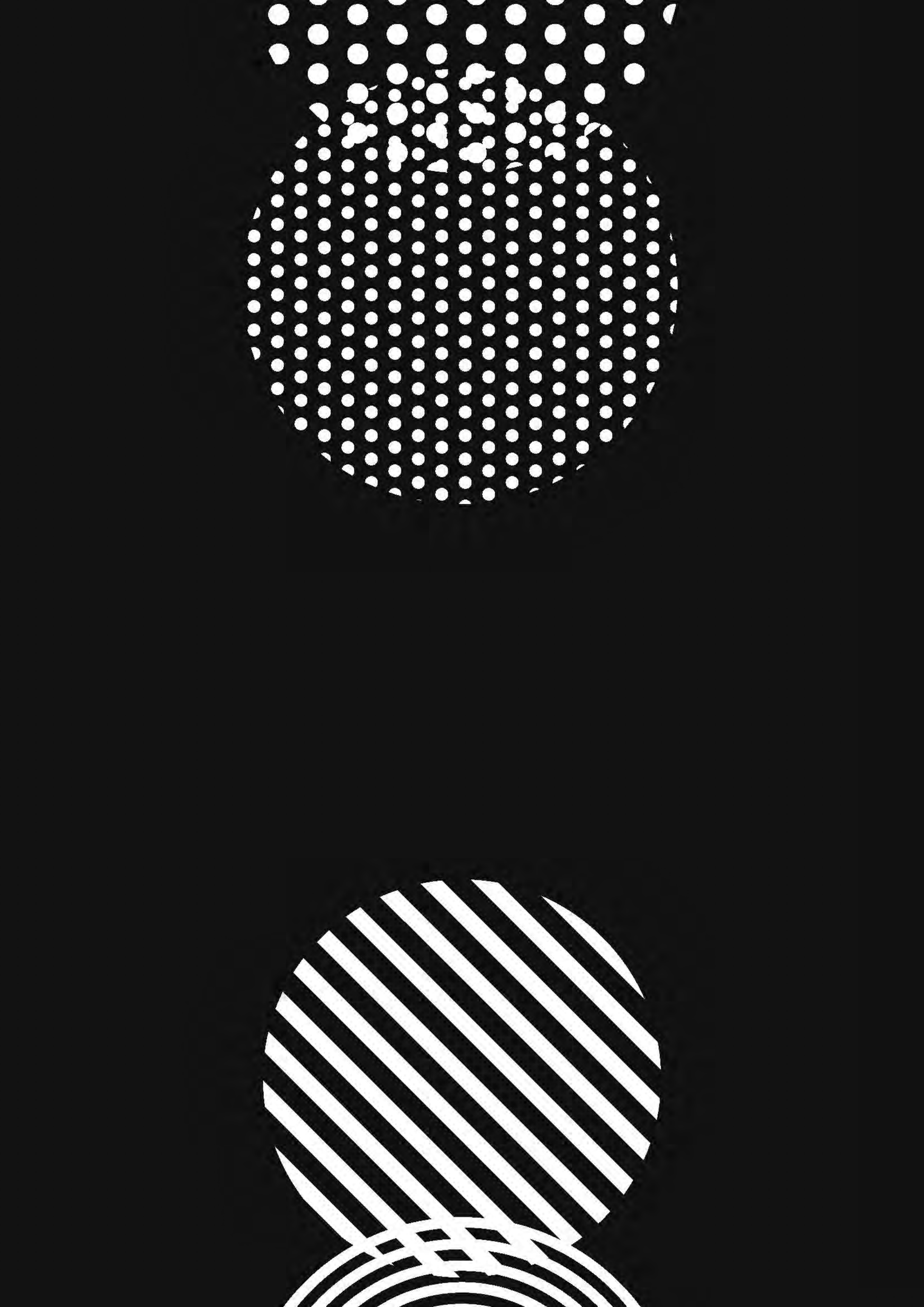

Differential forms

Def: Let M be an n-dim manifold. A differential p-form is a totally anti symmetric tensor of type $(0,p)$, i.e., $\omega_{a_1 \dots a_p}$ is a p-form if

$$\omega_{a_1 \dots a_p} = \omega_{[a_1 \dots a_p]}. \quad (1)$$

We denote the vec. sp. of p-forms at a point x by Λ^p_x and the collection of p-form fields by Λ^p.

remark: Note that $\Lambda^p_x = \{0\}$ if $p > n$ and $\dim \Lambda^p_x = \frac{n!}{p!(n-p)!}$ for $0 \leq p \leq n$.

Def: If we take the outerproduct of a p- and q-form we will get a tensor of type $(0, p+q)$; but since this tensor will not, in general, be totally anti-symmetric, it is not $(p+q)$-form. However, we can totally antisymmetrize this tensor, thus producing a map $\wedge : \Lambda^p_x \times \Lambda^q_x \to \Lambda^{p+q}_x$ via

$$(\omega \wedge \mu)_{a_1 \dots a_p b_1 \dots b_q} = \frac{(p+q)!}{p!\,q!} \omega_{[a_1 \dots a_p} \mu_{b_1 \dots b_q]} \quad (2)$$

Def: We define the vec. sp. of all differential forms at x to be the direct sum of the Λ^p_x:

$$\Lambda_x = \bigoplus_{p=0}^{n} \Lambda^p_x \quad (3)$$

remark: The map $\wedge : \Lambda_x \times \Lambda_x \to \Lambda_x$ gives Λ_x the structure of a Grassmann algebra over the vec. sp. of 1-forms.

Def: If we are given a derivative op., ∇_a, we could define a map from C^∞ p-form fields to $(p+1)$-form fields by:

$$\omega_{a_1 \dots a_p} \longrightarrow (p+1) \nabla_{[b} \omega_{a_1 \dots a_p]} \quad (4)$$

If instead we were given another der. op., $\tilde{\nabla}$, we would obtain the map

$$\omega_{a_1 \dots a_p} \longrightarrow (p+1) \tilde{\nabla}_{[b} \omega_{a_1 \dots a_p]} \quad (5)$$

$$\nabla_{[b} \omega_{a_1 \dots a_p]} - \tilde{\nabla}_{[b} \omega_{a_1 \dots a_p]} = \sum_{j=1}^{p} C^d{}_{[b a_j} \omega_{a_1 \dots d \dots a_p]} = 0 \quad (6)$$

since $C^c{}_{ab}$ is symmetric in a and b.

- The map in eq (4) is independent of der. op
- We denot this map by d.
- We may use ∂_a to calculate d.

remark: Since the index structure of diff. forms is trivial, it is customary to drop them, e.g. $\omega_{a_1 \dots a_p} \to \boldsymbol{\omega}$

$(\omega \wedge \mu)_{a_1 \dots b_q} \to \boldsymbol{\omega \wedge \mu}$

Def: We denote the $(p+1)$-form resulting from the action of the map $d: \Lambda^p \to \Lambda^{p+1}$ on the p-form $\boldsymbol{\omega}$ by $d\boldsymbol{\omega}$

An important prop. of the map d is that $d^2 = d \circ d = 0$. [This result known as the Poincaré lemma, follows from we can compute d using an ord. der. op.]

We have for an arbitrary C^∞ p-form $\boldsymbol{\omega}$:

$$(d^2\omega)_{bca_1 \dots a_p} = (p+2)(p+1)\partial_{[b}\partial_c \omega_{a_1 \dots a_p]} = 0 \quad (7)$$

because of equality if mixed partial dir. in $\mathbb{R}^n$

Remark: It can be shown that if one has a <u>closed</u> p-form, i.e., a p-form $\boldsymbol{\alpha}$ satisfying $d\boldsymbol{\alpha} = 0$, then locally this form is <u>exact</u>, i.e., there exists a $(p-1)$-form $\boldsymbol{\beta}$ such that $\boldsymbol{\alpha} = d\boldsymbol{\beta}$. However, in general this result is not valid globaly.

Integration

Def: ! M be an n-dim manifold. At each point $x \in M$, the vec. sp. of n-forms will be 1-dim. If it is possible to find a conticous, nowhere vanishing n-form field $\boldsymbol{\epsilon} = \epsilon_{[a_1 \dots a_n]}$ on M, the M said to be orientable and $\boldsymbol{\epsilon}$ is said to provide an orientation.

Proposition: Two orientations $\boldsymbol{\epsilon}$ & $\boldsymbol{\epsilon}'$ are considered equivalent if $\boldsymbol{\epsilon} = f\boldsymbol{\epsilon}$, where f is a (strictly) positive func., so any orientable manifold posesses 2 inequivalent orientations.

<u>left handed</u> or <u>right handed</u>

remark: It is easy to check that the manifolds $\mathbb{R}^m$ and S^m are orientable. Indeed it is not difficult to show that every simply connected manifold is orientable.

remark: The product of any 2 orientable manifold is orientable.

E.G.: Möbius strip $\to$ nonorientable manifold.

Integral of cont. n-form field $\boldsymbol{\alpha}$ over n-dim orientable manifold

Consider an open region $U \subset M$ covered by a single coord. syst. ψ. If we expand $\boldsymbol{\epsilon}$ in the coord. basis of ψ, we will obtain

$$\boldsymbol{\epsilon} = h\, dx^1 \wedge \dots \wedge dx^n \qquad (8)$$

where h is nonvanishing.

- $h > 0 \Rightarrow \psi$ is right handed
- $h < 0 \Rightarrow \psi$ is left handed

$\Big\}$ to $\boldsymbol{\epsilon}$

We may also expand $\boldsymbol{\alpha}$

$$\boldsymbol{\alpha} = a(x^1, \dots, x^n)\, dx^1 \wedge \dots \wedge dx^n \quad (9)$$

If ψ is right handed, we define the integral of $\boldsymbol{\alpha}$ over the region U by

$$\int_U \boldsymbol{\alpha} = \int_{\psi[U]} a\, dx^1 \dots dx^n \quad (10)$$

where the right hand side is the standard Reimann (or Lebesgue) integral in $\mathbb{R}^n$

remark: If ψ is left handed, we define $\int_U \boldsymbol{\alpha}$ to be minus the right hand side of eq (10)

Property: Integration is independent of the choice of coord. sys. ψ, covering U

$$\boldsymbol{\alpha} = a' \, dx'^1 \wedge \dots dx'^n \quad (11) \qquad a' = a \det\left(\frac{\partial x^\mu}{\partial x'^\nu}\right) \quad (12)$$

The standard law for transformation of integrals in $\mathbb{R}^n$ the shows that our definition, eq. (10), is coordinate independent.

To define the integral of $\boldsymbol{\alpha}$ over all M, we use the paracompactness prop. of M. A paracompact manifold can be covered by a countable collection $\{O_i\}$ of locally finite coord. neighborhoods such that each $\bar{O}_i$ is compact. Furthermore a partition of unity $\{f_i\}$ subordinate to this covering wil exist.

If $\sum_i \int_{\psi_i[O_i]} f_i |a_i|\, dx^1 \dots dx^n < \infty$, we say $\boldsymbol{\alpha}$ is integrable and we define

$$\int_M \boldsymbol{\alpha} = \sum_i \int_{O_i} f_i \, \boldsymbol{\alpha} \quad (13)$$

remark: It can be shown that this def. is independent of the choice of covering $\{O_i\}$ and partition of unity $\{f_i\}$ and thus properly defines $\int_M \boldsymbol{\alpha}$.

"Well behaved surface" $\underset{\text{Wald}}{\dot{N}} \equiv \partial N \quad int(N) = N^o$

Def: ! S be an manifold of dim $p<n$. If $\phi: S \to M$ is C^∞, is localy 1-1 and $\phi^{-1}: \phi[O] \to S$ is C^∞, then $\phi[S]$ is said to be immersed submanifold of M.

Def: If in addition, ϕ is globaly 1-1 (i.e., $\phi[S]$ doesn't "intersect itself"), then $\phi[S]$ is said to be an embeded submanifold of M.

We shall use the notion of embeded submanifold as our precise notion of "well behaved surface.

Def: An embeded submanifold of dimension $(n-1)$ is called a hypersurface.

Stokes's theorem:
If N is an orientable manifold with boundary, then an orientation on N induces a natural orientation on the boundary as follows:

1) Consider the coord. sys. on ∂N which arises from deleting, x^1, of the right-handed coord. sys. on N in the family of charts that makes N into a manifold with boundary

2.) Define an orientation on ∂N which makes the coord. sys. be right-handed. To do so:

(i) Verify that the Jacobian, $\det(\partial x^\mu/\partial x^{\mu'})$, is positive in the overlap region of any such 2 coord. sys.

(ii) Choose a partition func. of unity (F_i, U_i) of ∂N, where U_i is a coord. neighborhood of this type.

3) Define $\tilde{\epsilon}$ on ∂N by $\tilde{\epsilon} = \sum_i F_i \, dx_i^2 \cdots dx_i^n$ (14)

Then $\tilde{\epsilon}$ is continuous and nonvanishing and thus defines the desired orientation of ∂N.

Theorem (Stokes's theorem):

! N be an n-dim compact oriented manifold with boundary and let α an $(n-1)$-form on M which is C^1. Then

$$\int_{N^o} \boldsymbol{\alpha} = \int_{\partial N} \boldsymbol{\alpha} \quad (15) \qquad N^o\text{-interior of } N$$

Integration of func. on an orientable manifold M can be accomplished if one is given a volume element, that is, continuous nonvanishing n-form $\boldsymbol{\epsilon}$. The integral of f over M is defined by

$$\int_M f = \int_M f \boldsymbol{\epsilon} \quad (16)$$

remark: If one is given only the structure of a manifold M, there is no natural choice of volume element

If M has a metric, g_{ab} $\Rightarrow$ natural choice of $\boldsymbol{\epsilon}$ is defined up to a sign.

$$\epsilon^{a_1 \dots a_n} \epsilon_{a_1 \dots a_n} = (-1)^s n! \quad (17)$$

s - number of $(-)$ signs in g_{ab}

$$s = \begin{cases} 1 & \text{for Lorentz signature} \\ 0 & \text{for Reimann signature} \end{cases}$$

derivation with ∇_a associated with g_{ab}:

$$2\epsilon^{a_1 \dots a_n} \nabla_b \epsilon_{a_1 \dots a_n} = 0 \quad (18)$$

$$\underbrace{\nabla_b \epsilon_{a_1 \dots a_n}}_{} = 0 \quad (19)$$

totaly antisym. $\rightarrow$ last n indices and $\epsilon^{a_1 \dots a_n}$ is non-vanishing

$$\epsilon^{a_1 \dots a_n} \nabla_b \epsilon_{b_1 \dots b_n} = (-1)^s n! \, \delta^{[a_1}{}_{b_1} \cdots \delta^{a_n]}{}_{b_n} \quad (20)$$

$\delta^a{}_b$ - identity map on the tangent sp.

Eq. (20) follows from the fact that the tensors of type (n,n) on an n-dim. manifold which are totaly antisym. in all lower & upper indices form a 1-dim vec. sp. and thus must be proportional to the antisymmetrized product of $\delta^a{}_b$; the const. of proportionality is fixed by normalization cond. (eq. (17)).

Contraction of eq (20) over j of its indeces yields

$$\epsilon^{a_1 \dots a_j a_{j+1} \dots a_n} \epsilon_{b_1 \dots b_j b_{j+1} \dots b_n} = (-1)^s (n-j)! \, j! \, \delta^{[a_{j+1}}{}_{b_{j+1}} \cdots \delta^{a_n]}{}_{b_n} \quad (21)$$

$$\epsilon_{\mu_1 \dots \mu_n} = \begin{cases} (-1)^P & \text{if all } \mu_i \text{ are distinct} \\ 0 & \text{otherwise} \end{cases} \qquad \text{by eq. (17)}$$

P - signature of pertusmutation $(1 \dots n) \rightarrow (\mu_1 \dots \mu_n)$

In a coord. basis the comp. of $\boldsymbol{\epsilon}$ satisfy:

$$\underbrace{\sum_{\mu_1 \dots \nu_n} g^{\mu_1 \nu_1} \cdots g^{\mu_n \nu_n} \epsilon_{\mu_1 \dots \mu_n} \epsilon_{\nu_1 \dots \nu_n}}_{n! (\epsilon_{1 \dots n})^2 \cdot \det(g^{\mu\nu})} = (-1)^s n! \quad (22)$$

$$\epsilon_{1 \dots n} = [(-1)^s \underbrace{\det(g_{\mu\nu})}_{g}]^{1/2} = \sqrt{|g|} \quad (23)$$

$$\boldsymbol{\epsilon} = \sqrt{|g|} \, dx^1 \wedge \dots \wedge dx^n \quad (24)$$

Using the natural volume element $\boldsymbol{\epsilon}$ associated with the metric, we can convert Stokes's theorem, eq (15), into "Gauss's law" form.

! N be an oriented, compact n-dim manifold with boundary. ! g_{ab} be a metric on N with associated volume element $\boldsymbol{\epsilon}$. Given any C^1 vec. field v^a, we obtain an $(n-1)$-form $\boldsymbol{\alpha}$ by

$$\alpha_{a_1 \ldots a_{n-1}} = \epsilon_{b a_1 \ldots a_{n-1}} v^b \quad (25)$$

$$(d\alpha)_{c a_1 \ldots a_{n-1}} = n \nabla_{[c} \left(\epsilon_{|b| a_1 \ldots a_{n-1}]} v^b \right)$$

$$= n\, \epsilon_{b[a_1 \ldots a_{n-1}} \nabla_{c]} v^b \quad (26) \quad / \text{eq. } (19)$$

On the other hand any totally antisym. tensor of type $(0,n)$ must be proportional to $\boldsymbol{\epsilon}$:

$$\epsilon_{b[a_1 \ldots a_{n-1}} \nabla_{c]} v^b = h\, \epsilon_{c a_1 \ldots a_{n-1}} \quad (27)$$

$$\nabla_b v^b = nh \qquad d\boldsymbol{\alpha} = (\nabla_a v^a)\boldsymbol{\epsilon}$$

and thus Stokes's theorem states that

$$\int_{N^\circ} \nabla_a v^a = \int_{\partial N} \epsilon_{b a_1 \ldots a_{n-1}} v^b \quad (28)$$

The right-hand side can be expressed as follows. The metric on g_{ab} on N induces a tensor field h_{ab} on ∂N by restriction of g_{ab} to vec. tangent to ∂N. If h_{ab} is nondegenerate (∂N is not a null surface) we may use it to define a volume element $\tilde{\boldsymbol{\epsilon}}$ on ∂N.

$$\frac{1}{n} \epsilon_{a_1 \ldots a_n} = n_{[a_1} \tilde{\epsilon}_{a_2 \ldots a_n]}$$

n^b is the unit normal to ∂N an is chosen to be:

- outward pointing if spacelike
- inward pointing if timelike

$\Big\}$ in order to $\tilde{\boldsymbol{\epsilon}}$ be of the orientation class used in Stokes's theorem.

$$\epsilon_{b a_1 \ldots a_{n-1}} v^b = (n_b v^b)\, \tilde{\epsilon}_{a_1 \ldots a_{n-1}}$$

$$\int_{N^\circ} \nabla_a v^a = \int_{\partial N} n_a v^a \qquad \forall\, v^a - C^1 \text{ vec. field}$$

Frobenius's Theorem

! M be n-dim. manifold. An issue which arises frequently is the following:

At each point $x \in M$ we are given a subsp. $W_x \subset V_x$ of the tangent sp V_x with $\dim W_x = m < n$. W_x required to vary smoothly with x in the sense that $\forall x \in M$ we can find an open neighborhood O of x such that in O, W is spanned by C^∞ vectors. ($W = \{W_x\}$)

We wish to know whether we can find intedral submanifolds of W.

remark: An important special case of this general problem arises when one has a metric on M and wishes to know if a vec. field ξ^a is orthogonal to a family of hypersurfaces, i.e., whether the $(n-1)$-dim subsp., W, orthogonal to ξ^a are integrable

If $\dim W = 1 \Longrightarrow$ finding integral curves of C^∞ v^a.

If $\dim W > 1$, it is possible for the W-planes to "twist around" so that integral integral submanifolds cannot be found.

If we could find int. subman., we could span W in a neighborhood of any point by coord. vec. fields $\{(X_1^a, \dots, X_m^a) \in M : [X_\mu, X_\nu] = 0\}$.

$\forall Y^a, Z^a \in W$:

$$[Y, Z] = \sum_{\mu,\nu} [f_\mu X_\mu, g_\nu X_\nu] = \sum_{\mu,\nu} (f_\mu X_\mu(g_\nu) - g_\mu X_\mu(f_\nu)) X_\nu \in W$$

Def: If W satisfies the prop. that $[Y, Z]^a \in W$ $\forall Y^a, Z^a \in W$, then W is said to be involute.

Theorem (Frobenius's theorem):

A necessary and sufficient condition for a C^∞ specification, W, of m-dim subsp. of the tangent sp. at each $x \in M$ to possess integral submanifolds is that W be involute, i.e., $\forall Y^a, Z^a \in W$ we have $[Y, Z]^a \in W$.