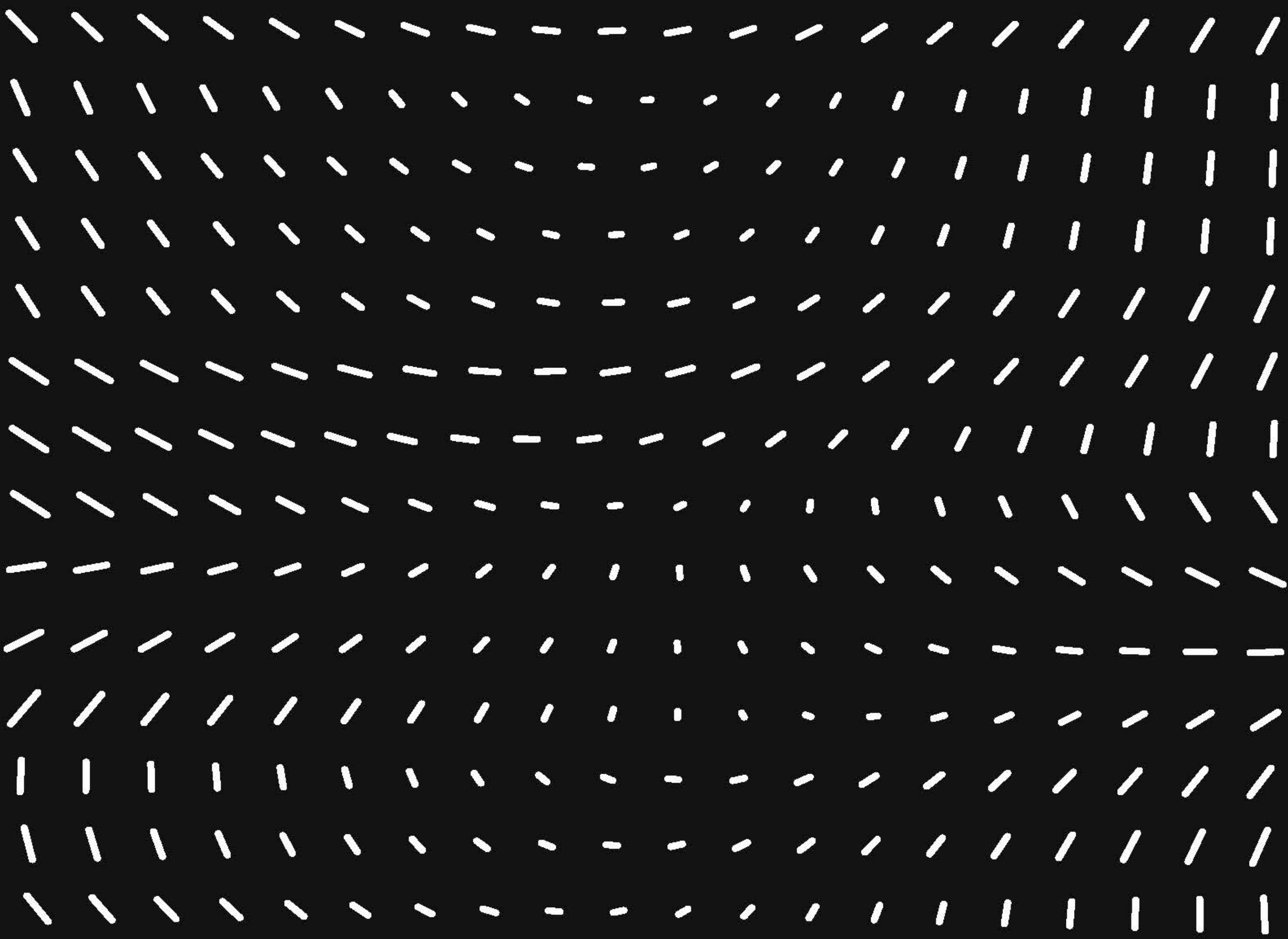

Maps of manifolds

Let M and N be manifolds and let $\phi: M \to N$ be a C^∞ map.

1.) In "natural manner" ϕ "pulls back" a func. $f: N \to \mathbb{R}$ on N to the func $f\circ\phi: M \to \mathbb{R}$ obtained by composing f with ϕ

2) ϕ "carries along" tangent vec at $p \in M$ to tangent vec at $\phi(p) \in N$
↳ it defines a map $\phi^*: V_p \to V_{\phi(p)}$ → For $v \in V_p$ we define $\phi^* v \in V_{\phi(p)}$ by

$$(\phi^* v)(f) = v(f\circ\phi) \quad \forall f: N \to \mathbb{R}$$

where f is C^∞

Note: ϕ^* is linear and may be viewed as the "derivative of ϕ" at p.

Remark: The matrix of components of ϕ^* in the coord. basis of a coord. sys $\{x^\nu\}$ at p and a coord. sys. $\{y^\mu\}$ at $\phi(p)$ equals the Jacobian mtx of the map ϕ between the coord., i.e., $(\phi^*)^\mu{}_\nu = \partial y^\mu / \partial x^\nu$.

By the implicit func. theorem, $\phi: M \to N$ will be one-to-one in a neighborhood of p if $\phi^*: V_p \to V_{\phi(p)}$ is one-to-one.

3.) We can use ϕ to "pull back" dual vec. at $\phi(p)$ to dual vec. of p. We define the map $\phi_*: V^*_{\phi(p)} \to V^*_p$ by requiring that $\forall v^a \in V_p$

$$(\phi_* \mu)_a v^a = \mu_a (\phi^* v)^a$$

We can extend the action of ϕ_* to map tensors of type $(0,l)$ at $\phi(p)$ to tensors of type $(0,l)$ at p by

$$(\phi_* T)_{a_1 \dots a_l} (v^1)^{a_1} \dots (v^l)^{a_l} = T_{a_1 \dots a_l} (\phi^* v^1)^{a_1} \dots (\phi^* v^l)^{a_l}$$

Similarly for tensors of type $(k,0)$

$$(\phi^* T)^{b_1 \dots b_k} (\mu_1)_{b_1} \dots (\mu_k)_{b_k} = T^{b_1 \dots b_k} (\phi_* \mu_1)_{b_1} \dots (\phi_* \mu_k)_{b_k}$$

Remark: We cannot extend ϕ^* or ϕ_* to mixed tensors since ϕ^* doesn't know how to 'carry along' lower index tensors, while ϕ_* doesn't know how to "pull back" upper index tensors.

A C^∞ map $\phi: M \to N$ is said to be a diffeomorphism if it is one-to-one, onto, and its inverse is C^∞. If ϕ is diffeomorphism ⇒ we can use ϕ^{-1} to extend the definition of ϕ^* to tensors of all types by using the fact that $(\phi^{-1})^*$ goes from $V_{\phi(p)}$ to V_p. If $T^{b_1 \dots b_k}{}_{a_1 \dots a_l}$ is a tensor of type (k,l) at p, we define the tensor $(\phi^* T)^{b_1 \dots b_k}{}_{a_1 \dots a_l}$ at $\phi(p)$ by

$$(\phi^* T)^{b_1 \dots b_k}{}_{a_1 \dots a_l} (\mu_1)_{b_1} \dots (\mu_k)_{b_k} (t^1)^{a_1} \dots (t^l)^{a_l} = T^{b_1 \dots b_k}{}_{a_1 \dots a_l} (\phi_* \mu_1)_{b_1} \dots (\phi_* \mu_k)_{b_k} [(\phi^{-1})^* t^1]^{a_1} \dots [(\phi^{-1})^* t^l]^{a_l}$$

Remark: Similarly, we could extend the map ϕ_* to all tensors. However, it is not difficult to show that $\phi_* = (\phi^{-1})^*$.

If $\phi: M \to M$ is diffeomorphism and T is a tensor field on M, we can compare T with $\phi^* T$. If $\phi^* T = T \Rightarrow$ "moved" T via ϕ and "stayed the same" $\to \phi$ is a symmetry transformation for T.

Remark: For the metric g_{ab} a symmetry trans. is called an isometry.

Remark: If $\phi: M \to N$ is a diffeomorphism $\Rightarrow$ M & N have identical manifold structure.

If a theory describes nature in terms of a space-time manifold, M, and tensor fields, $T^{(i)}$, defined on the manifold, then if $\phi: M \to N$ is a diffeomorphism, the solutions $(M, T^{(i)})$ and $(N, \phi^* T^{(i)})$ have physically identical properties.

Lie derivative

Let M be a manifold and let ϕ_t be a one-parameter group of diffeomorphisms. ϕ_t will be generated by a vectorfield, v^a, and ϕ_t^* carries along C^∞ tensorfield $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$. Comparison of $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ and $\phi_{-t}^* T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ for small t gives rise to the notion of Lie derivative, $\mathcal{L}_v$, with respect to v^a. More precisely:

$$\mathcal{L}_v T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = \lim_{t \to 0} \left\{ \frac{\phi_t^* T^{a_1 \dots a_k}{}_{b_1 \dots b_l} - T^{a_1 \dots a_k}{}_{b_1 \dots b_l}}{t} \right\}$$

where all tensors appearing are evaluated at the same point p.

Note: index on v^a is droped in the symbol $\mathcal{L}_v$ since it presence could lead to confusion.

Properties of Lie derivative:

(i) ! $\mathcal{T}(k,l)$ be a group of C^∞ type (k,l) tensor fields. $\mathcal{L}_v: \mathcal{T}(k,l) \to \mathcal{T}(k,l)$ is a linear map.

(ii) $\mathcal{L}_v$ satisfies the Leibnitz rule on outer products of tensors

$$\mathcal{L}_v (T, R) = T \cdot \mathcal{L}_v R + R \cdot \mathcal{L}_v T$$

(iii) Since v^a tangent to the integral curves of ϕ_t, for func. $f: M \to \mathbb{R}$ we have

$$\mathcal{L}_v (f) = v(f)$$

(iv) $\mathcal{L}_v$ commutes with the contraction

(v) $\mathcal{L}_v T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = 0$ everywhere IFF $\forall t: \phi_t$ is a symmetry trans. for T.

The components of $\mathcal{L}_v$ of $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ in a coord. sys. adapted to v^a are simply

$$\mathcal{L}_v T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l} = \frac{\partial T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}}{\partial x^1} . \quad (*)$$

Thus in particular, ϕ_t will be a symmetry trans. of T IFF the components $T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$ in a coord. sys. adapted to v^a are independent of the integral curve coordinate x^1.
We can can obtain coord. independent expression for $\mathcal{L}_v$ of a vec. field w^a by noting that in an adapted coord. sys. we have by eq. $(*)$

$$\mathcal{L}_v w^\mu = \frac{\partial w^\mu}{\partial x^1}$$

Since $v^a = (\partial/\partial x^1)^a$ and $w^a = \sum_\mu w^\mu (\partial/\partial x^\mu)^a$, the commutator of v^a and w^a:

$$[v, w]^\mu = \sum_\nu \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right) = \frac{\partial w^\mu}{\partial x^1}$$

$$\mathcal{L}_v w^a = [v, w]^a$$

coord. sys independent formula for Lie derivative

The action of Lie derivative on all other types of tensor fields is determined by prop. (iii), and $\mathcal{L}_v w^a = [v, w]^a$, and the Leibnitz rule.

<u>E.G.:</u> for dual vec. field, μ_a, we have by prop. (iii)

$$\mathcal{L}_v (\mu_a w^a) = v(\mu_a w^a)$$

where w^a is an arbitrary field. By the Leibnitz rule and $\mathcal{L}_v w^a = [v, w]^a$, we have

$$\mathcal{L}_v (\mu_a w^a) = w^a \mathcal{L}_v \mu_a + \mu_a [v, w]^a$$

If D_a is a der. op. on M, we have by prop. (4) and (2) of the definition der. op

$$v(\mu_a w^a) = v^b D_b (\mu_a w^a) = v^b w^a D_b \mu_a + v^b \mu_a D_b w^a$$

$$[v, w]^a = v^b D_b w^a - w^b D_b v^a$$

$$v^b w^a D_b \mu_a + v^b \mu_a D_b w^a = w^a \mathcal{L}_v \mu_a + \mu_a v^b D_b w^a - \mu_a w^b D_b v^b$$

$$\mathcal{L}_v \mu_a = v^b D_b \mu_a + \mu_b D_a v^b$$

More generaly for an arbitrary tensor field:

$$\mathcal{L}_v T^{a_1 \dots a_k}{}_{b_1 \dots b_l} = v^c D_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} - \sum_{i=1}^{k} T^{a_1 \dots c \dots a_k}{}_{b_1 \dots b_l} D_c v^{a_i} + \sum_{j=1}^{l} T^{a_1 \dots a_k}{}_{b_1 \dots c \dots b_l} D_{b_j} v^c$$

If $\phi: M \to M$ diffeomorphism $\Rightarrow$ (M, g_{ab}) and $(M, \phi^* g_{ab})$ represent the same physical space-time. If we consider 1-parameter family of space-times $(M, g_{ab}(\lambda))$ and $(M, \phi^*_\lambda g_{ab}(\lambda))$ represent the same 1-parameter family, where ϕ^*_λ is an arbitrary 1-parameter group of diffeomorphisms.

If we consider the 1st order perturbation of $g_{ab}|_{\lambda=0}$ we obtained by differentiating $g_{ab}(\lambda)$ wit respect to λ at $\lambda=0$, we find that $\gamma_{ab} = dg_{ab}/d\lambda|_{\lambda=0}$ and $\gamma_{ab} = d(\phi^*_\lambda g_{ab})/d\lambda|_{\lambda=0}$ represent the same perturbation.

$$\gamma'_{ab} = \gamma_{ab} - \mathcal{L}_v g_{ab}$$

v^a is the vec. field which generates ϕ^*_λ and $g_{ab} = g_{ab}(\lambda=0)$. Thus, the gauge freedom in perturbation, γ_{ab}, is given by $\mathcal{L}_v g_{ab}$, where v^a is an arbitrary vec. field.

$$\mathcal{L}_v g_{ab} = v^c \nabla_c g_{ab} + g_{cb} \nabla_a v^c + g_{ac} \nabla_b v^c$$
$$= \nabla_a v_b + \nabla_b v_a$$

Thus, <u>the gauge transformation</u> of linearized GR. about a solution g_{ab} are

$$\gamma_{ab} \longrightarrow \gamma'_{ab} = \gamma_{ab} - \nabla_a v_b - \nabla_b v_a$$

This is closly analogous to the gauge freedom $A_a \rightarrow A'_a - \nabla_a \chi$ of electro-magnetism.

Killing Vector Fields

If $\phi_t: M \longrightarrow M$ is 1-parameter group of isometries, $\phi^*_t g_{ab} = g_{ab}$, the vec. field ξ^a which generates ϕ_t is called a Killing vec. field. The necessary and sufficient condition for ϕ_t to be a group of isometries is $\mathcal{L}_v g_{ab} = 0$. Thus, according to $\mathcal{L}_v g_{ab} = \nabla_a v_b - \nabla_b v_a$ the necessary and sufficient cond. tha ξ^a be a Killing vec. field is that it satisfy Killing's equation:

$$\nabla_a \xi_b + \nabla_b \xi_a = 0$$

<u>Proposition</u>: ! ξ^a be a Killing vec. field and let γ be a geodesic with tangent u^a. Then $\xi_a u^a$ is constant along γ.

<u>Proof</u>: We have

$$u^b \nabla_b (\xi_a u^a) = u^b u^a \nabla_b \xi_a + \xi_a u^b \nabla_b u^a = 0$$

since the first term vanishes by Killing's equation and the second term vanishes by the geodesic equation. $\square$

Since in GR timelike geodesic represent the spacetime motions of freely falling particles and null geodesic represent the paths of light rays, this prop. can be interpreted as saying that every 1-parameter family symmetries gives rise to a conserved quantity for particles and light rays. This conserved quantity enables one to determine the gravitational redshift in stationary spacetimes and is extremely useful for integrating the geodesic eq. when symmetries are present.

The Reimann tensor:

$$\nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c = R_{abc}{}^{d} \xi_d \quad / \text{Killing eq.}$$

$$\nabla_a \nabla_b \xi_c + \nabla_b \nabla_c \xi_a = R_{abc}{}^{d} \xi_d$$

If we write down the cyclic permutation of indices $(abc) \Rightarrow$ adding them together, we get:

$$2 \nabla_b \nabla_c \xi_a = (R_{abc}{}^{d} + R_{bca}{}^{d} - R_{cab}{}^{d}) \xi_d = -2 R_{cab}{}^{d} \xi_d$$

We used the symmetry properties of $R_{abc}{}^{d}$. Thus, $\forall \xi^a$ we obtain the formula:

$$\nabla_a \nabla_b \xi_c = - R_{bca}{}^{d} \xi_d$$

A Killing field, ξ^a, is complitly determined by the values of ξ^a and $L_{ab} \equiv \nabla_b \xi_a$ at any point $p \in M$.

If we are given (ξ^a, L_{ab}) at $p \Rightarrow (\xi^a, L_{ab})$ at any other point q is determined by integration of the system of ODE

$$v^a \nabla_a \xi_b = v^a L_{ab}$$

$$v^a \nabla_a L_{bc} = - R_{bca}{}^{d} \xi_d v^a$$

along the curve connecting p and q, where v^a denotes the tangent to the curve

Corollaries of this result

(i) if a Killing field and its derivative vanish at a point, then the Killing field vanishes everywhere

(ii) on a manifold of dim n, ther can be at most

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

linearly independent Killing fields, since this is the dim of the space of initial data for (ξ^a, L_{ab}).

If we contract over a and b, we find

$$\nabla^a \nabla_a \xi_c = - R_c{}^{d} \xi_d$$

Thus, in a vacuum space time, $R_c{}^{d} = 0$, ξ^a satisfies the src. free Maxwell eq. for a vec. potential in the Lorentz gauge.

The Lorentz gauge condition $D_a \xi^a = 0$.