INTRODUCTION TO GENERAL RELATIVITY

Geodesics

GEODESICS IN GEOGRAPHY AND EARTH SCIENCES

A little bit of cultural history…

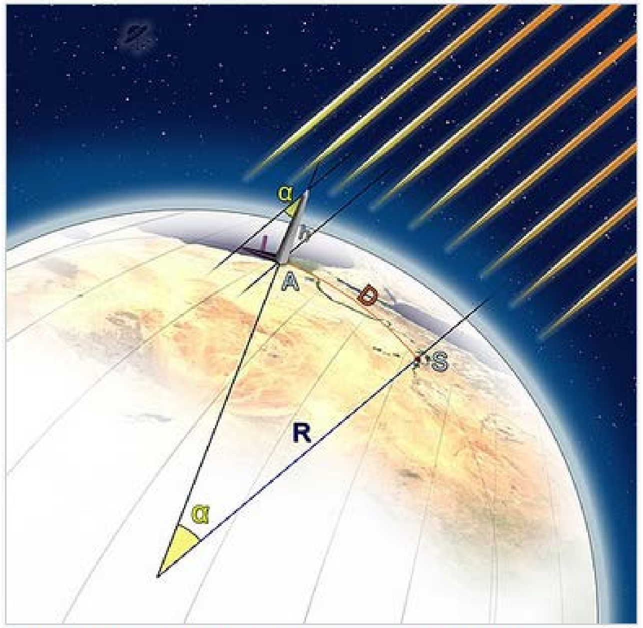
Eratosthenes

Born	276 BC Cyrene
Died	194 BC (around age 82) Alexandria
Occupation	Scholar Librarian Poet Inventor
Known for	Sieve of Eratosthenes Founder of Geography

- The noun "*geodesy*" and the adjective "*geodesic*" come from the Ancient Greek word γεωδαισία (literally, "division of Earth").

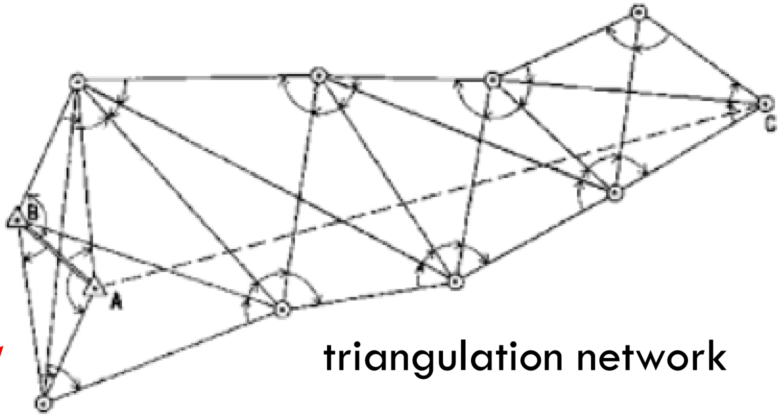
- *Eratosthenes of Cyrene* (c. 276–194 BC) estimated the value of the Earth's circumference by using the extensive survey results he could access in his role at the Library of Alexandria.

- He calculated that the circumference has a length of 252,000 stadia by measuring distances between two Egyptian cities, Alexandria and Syene (modern Aswan) at "local noon" on the summer solstice. → Since 1 Egyptian stadium is equal to 157.5 meters, the result is 39,375 km, which is 1.4% less than the real number, 40,076 km.



Measure of Earth's circumference according to Cleomedes' simplified version, based on the wrong assumption that Syene is on the Tropic of Cancer and on the same meridian as Alexandria

A geodesist determines the location of a point by measuring only angles → **triangulation networks**



triangulation network

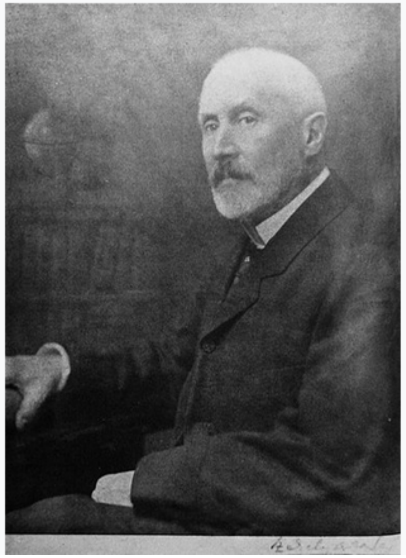
Besides geographical surveys for *more detailed* **maps** and **charts**…

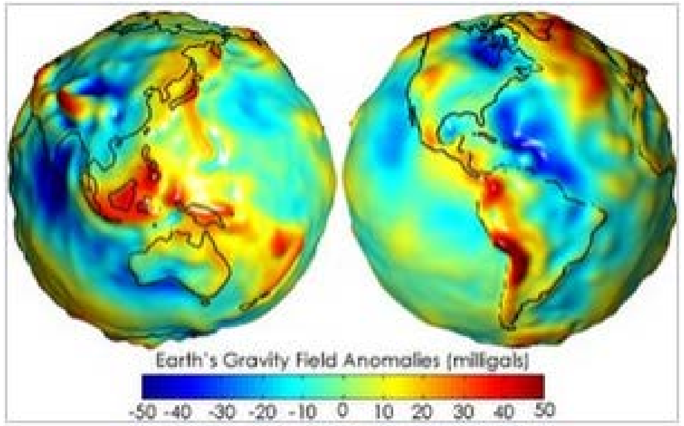
Geodesy is primarily concerned with positioning within the temporally varying gravitational field:

- *accurately measuring* and understanding Earth's geometric shape, orientation in space and **gravitational field**

Eötvös pendulum

Loránd Eötvös





Three-dimensional visualization of gravity anomalies in units of Gal., using pseudo color and shaded relief with vertical exaggeration.

GEODESICS IN GENERAL RELATIVITY

In general relativity, a geodesic generalizes the notion of a "straight line" to curved spacetime. → They are lines that "curve as little as possible", i.e.. they are the "straightest possible lines" in a curved geometry.

Definition

Given a derivative operator, ∇_a, we define *geodesic* to be a curve whose tangent vector is parallel propagated along itself, i.e. a curve whose tangent, T^a, satisfies the equation

$$T^a \nabla_a T^b = 0.$$

- To have a curve „as straight as possible”, one the tangent vector (to a curve point) is required to point in the same direction when parallel transported, but *it does not have to maintain the same length*. This yields the **weaker condition**:

$$T^a \nabla_a T^b = \alpha T^b$$

where α is an arbitrary function on the curve.

- Given a curve that satisfies the *weaker condition*, one can always reparametrize it so that it satisfies the *stronger condition*.

Definition

A parametrization which yields the equation for the stronger condition is called an *affine parametrization*.

→ Thus, our definition of geodetics requires it to be affinely parametrized.

- To get some insight into the nature of the geodesic equation, let's write out the components of this equation in a coordinate basis!

The components T^μ of T^a in this coordinate basis satisfy

$$\frac{dT^\mu}{dt} + \sum_{\sigma,\nu} \Gamma^\mu{}_{\sigma\nu} T^\sigma T^\nu = 0.$$

However, for the components $T^\mu = dx^\mu/dt$, the geodesic equation becomes

$$\frac{d^2x^\mu}{dt^2} + \sum_{\sigma,\nu} \Gamma^\mu{}_{\sigma\nu} \frac{dx^\sigma}{dt}\frac{dx^\nu}{dt} = 0.$$

← A coupled system of n 2nd-order ODE for n functions $x^\mu(t)$.

Theorem

From the theory of ODEs, it is known that a unique solution always exists for any given initial value of dx^μ and dx^μ/dt. This means that *given* $p \in M$ *and any tangent vector,* $dx^\mu \in V_p$, *there always exists a unique geodesic through* p *with tangent* T^a.

Let's notice that the solution of the equation of motion in ordinary mechanics share this property: Given an initial position and velocity, a unique solution exists.

→ The ***existence*** and ***uniqueness*** of geodetics allows us to use them to construct coordinate systems that are very convenient for some computation.

Definition

Let $p \in M$ and let a map, called the *exponential map*, from the tangent space V_p to M be defined by mapping each $T^a \in V_p$ into the point in M lying at unit affine parameter from p along the geodesic through p with tangent T^a.

- For large T^a one might encounter a singularity before the affine parameter $t = 1$ is reached.
- Also geodesics may cross, thereby making the exponential map fail to be one-to-one. However, one can show that *there always exists a (sufficiently small) neighborhood* of the origin of V_p *on which the exponential map is defined and is one-to-one*.

- Since V_p is a n-dimensional vector space we may identify it with $\mathbb{R}^n$:

Definition

The use of exponential maps give us a coordinate system, called *Riemannian normal coordinates at p*.

- These coordinates have the property that all geodesics through p get mapped into straight lines through the origin of $\mathbb{R}^n$.
- Also the Christoffel symbol components $\Gamma^{\mu}{}_{\sigma\nu}$ vanish at p. $\longrightarrow$ Therefore, these coordinates are particularly useful if one is performing calculations at a given point.

Definition

In case the derivative operator ∇_a arises from a metric tensor g_{ab} a second type of coordinate system, called *Gaussian normal coordinates*, or *synchronous coordinates*, often is useful for calculations in situations where one is given a hypersurface S, i.e., an $(n-1)$-dimensional embedded submanifold of the n-dimensional manifold M.

At each point $p \in S$, the tangent space V_p of the manifold S can be viewed as an $(n-1)$-dimensional subspace of the tangent space V_p of M.

Theorem

There is a vector $n^a \in V_p$, unique up to scaling, which is orthogonal to all vectors in V_p. This vector, n^a, is said to be normal to S.

- In the case of a Riemannian metric, n^a cannot lie in V_p;
- In the case of a metric of indefinite signature, n^a could be a null vector, $g_{ab}n^a n^b = 0$, in which case it does lie in V_p and S is said to be a *null hypersurface* at point p.
- If S is nowhere null, we may normalize n^a by the condition $g_{ab}n^a n^b = \pm 1$.

For all $p \in S$ we construct the unique geodesic through p with tangent n^a. We choose arbitrary coordinates $(x^1, \dots, x^{n-1})$ on a portion of S and label each point in a neighborhood of S by the parameter t along the geodesic on which it lies and the coordinates $(x^1, \dots, x^{n-1})$ of the point $p \in S$ from which the geodesic emerge.

Properties of geodesics: orthogonality

Gaussian normal coordinates satisfy the important property that ***the geodesics remain orthogonal to all hypersurfaces*** S_t defined by the equation $t = const$.

For this property, it is enough to show that the geodesic tangent field n^a remains orthogonal to all of the coordinate basis fields $X_1^a, \dots, X_{n-1}^a$ which generate the tangent space to S_t. Denoting by X^a any one of these fields, we have

geodesic eq. | *n^a and X^a commute – being elements of a coordinate basis on M* | *Leibniz rule*

$$n^b \nabla_a (n_a X^a) = n_a n^b \nabla_b X^a = n_a X^b \nabla_b n^a = \frac{1}{2} X^b \nabla_b (n^a n_a) = 0,$$

where the last equality follows from the fact that the normalization condition $n^a n_a = \pm 1$ on S is *preserved by parallel transport* so that $n^a n_a$ is ***constant*** on M. Since $n_a X^a = 0$ on S, this equation shows that this condition is preserved.

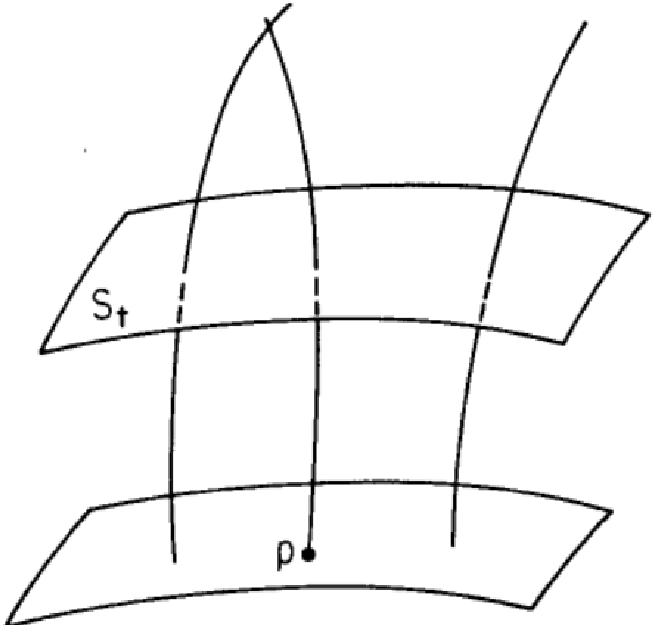


Fig. 3.4. The construction of Gaussian normal coordinates starting from the hypersurface S. The geodesics orthogonal to S eventually may cross, but until they do, Gaussian normal coordinates are well defined and the surfaces, S_t, of constant t remain orthogonal to the geodesics.

Properties of geodesics: extremal length

A further property of geodesics of a derivative operator arising from a metric is that ***they extremize the length of curves*** connecting given points as measured by the metric.

Definition

For a smooth curve C on a manifold M with Riemannian metric g_{ab}, the *length* l of C is defined by

$$l = \int \sqrt{g_{ab}T^aT^b}\,dt,$$

where T^a is the tangent to C and t is the *curve parameter*.

Definition

For a metric of Lorentz signature $(- \; + \; + \; \ldots \; +)$, a curve is said to be

- *Timelike* if the norm of its tangent is everywhere negative, $g_{ab}T^aT^b < 0$;
- *Null* if the norm of its tangent is everywhere zero, $g_{ab}T^aT^b = 0$;
- *Spacelike* if the norm of its tangent is everywhere positive, $g_{ab}T^aT^b > 0$.

- For spacelike curves, the length may again be defined with the above equation;
- For null curves the length be zero;
- For timelike curves, rather than the length, ***we change the sign in the square root*** and ***use the term*** *proper time*,

$$t = \int \sqrt{-g_{ab}T^aT^b}\, dt.$$

The length of curves which change from timelike to spacelike is not defined.

Notion

The *length* (or *proper time*) of a curve ***does not*** depend on the way in which the curve is parametrized! If we define a new parametrization $s = s(t)$, the new tangent will be $S^a = (dt/ds)T^a$ and the new length will be

$$l' \equiv \int \sqrt{g_{ab}S^aS^b}\, dt = \int \sqrt{g_{ab}S^aS^b}\, \frac{dt}{ds} ds = l.$$

Extremization condition

Let's derive the condition on a curve which makes it extemize the the length between endpoints! → Let's find those curves ***whose length does not change*** to first order under an arbitrary smooth deformation which keeps the end points fixed.

For definiteness, we consider a spacelike curve. Writing the equation of proper length in the coordinate basis yields

$$l = \int_a^b \sqrt{g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}}\, dt$$

where $C(a) = p$ and $C(b) = q$ are the endpoints of the curve. The extremization problem for l is mathematically identical to the extremization problem for the action in *Lagrangian mechanics*. The variation in l is

$$\delta l = \int_a^b \sqrt{\sum_{\mu,\nu} g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}} \sum_{\alpha,\beta} \left\{ g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{d(\delta x^\beta)}{dt} + \frac{1}{2} \sum_\sigma \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \delta x^\sigma \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right\} dt.$$

Without loss of generality, we may assume that the original curve was parametrized so that

$$g_{ab} T^a T^b = 1 = \sum_{\mu,\nu} g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}.$$

← Since the length is independent of the parametrization

With this choice of parametrization, the extremization condition is

$$0 = \int_a^b \sum_{\alpha,\beta} \left\{ g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{d(\delta x^\beta)}{dt} + \frac{1}{2} \sum_\sigma \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \delta x^\sigma \right\} dt$$

$$= \int_a^b \sum_{\alpha,\beta} \left\{ -\frac{d}{dt} \left(g_{\alpha\beta} \frac{dx^\alpha}{dt} \right) + \frac{1}{2} \sum_\lambda \frac{\partial g_{\alpha\lambda}}{\partial x^\sigma} \frac{dx^\alpha}{dt} \frac{dx^\lambda}{dt} \right\} \delta x^\beta \, dt\,.$$

No boundary terms occur in the integration by parts since δx^β vanishes at the endpoints.

This equation will hold for arbitrary δx^β if and only if

$$-\sum_{\alpha,\beta} g_{\alpha\beta} \frac{d^2 x^\alpha}{dt^2} - \sum_{\alpha,\lambda} \frac{\partial g_{\alpha\beta}}{\partial x^\lambda} \frac{dx^\alpha}{dt} \frac{dx^\lambda}{dt} + \frac{1}{2} \sum_{\alpha,\lambda} \frac{\partial g_{\alpha\lambda}}{\partial x^\beta} \frac{dx^\alpha}{dt} \frac{dx^\lambda}{dt} = 0.$$

Using our formula for $\Gamma^\sigma{}_{\alpha\lambda}$, and the equation of proper length, we see that the last equation is just the geodesic equation. ***Thus a curve extremizes the length between its endpoints if and only if it is a geodesic.***

An identical derivation shows that the curve which extremize proper time between two points are precisely the timelike geodesics. These derivations also show that the geodesic (with affine parametrization) can be obtained from variation of the Lagrangian:

$$L = \sum_{\mu,\nu} g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}\,.$$

In many cases, the *most efficient way* of computing the Christoffel symbol $\Gamma^{\mu}{}_{\sigma\nu}$ in a given coordinate basis is:

1) To start with the Lagrangian,
2) Write down the corresponding Euler-Lagrange equation,
3) And read off $\Gamma^{\mu}{}_{\sigma\nu}$ by comparison with the geodetic equation, $\frac{d^2x^{\mu}}{dt^2} + \sum_{\sigma,\nu} \Gamma^{\mu}{}_{\sigma\nu} \frac{dx^{\sigma}}{dt}\frac{dx^{\nu}}{dt} = 0.$

Properties of extremal length

- On a manifold with a Riemannian metric, one can always find curves of arbitrarily long length connecting any two points.
- However, the length will be bounded from below, and the curve of shortest length connecting two points is necessarily an extremum of length and thus a geodesic.

→ Thus, the shortest path between two points is always a "*straightest possible path*".

Note: ***A given geodesic connecting two points isn't necessarily the shortest path between them.***

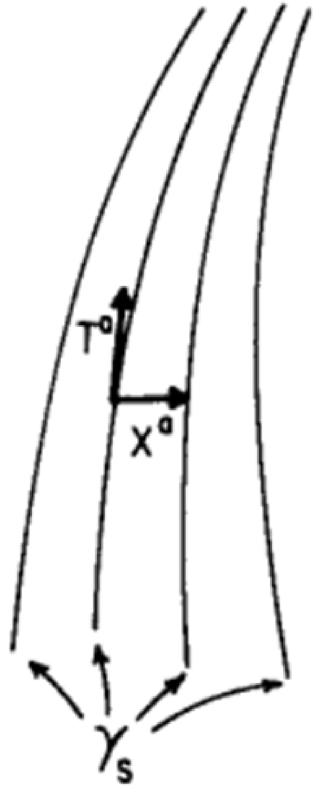
- For two given point that can be connected by a timelike curve, one can always find timelike curves of *arbitrarily small proper time* connecting the points.
- In some spacetimes the proper time of timelike curves connecting the two given points *need not be bounded from above*; but *if a curve of greatest proper time exists*, ***it must be a time like geodesic***.

Our final task is to derive the geodesic equation, the equation which relates the tendencies of geodesics to accelerate towards or away from each other to the curvature of the manifold.

Definition

Let $\gamma_s(t)$ denote a smooth one parameter family of geodesics, that is for each element $s \in \mathbb{R}$, the curve γ_s is a geodesic (parametrized by affine parameter t) and the map $(t, s) \to \gamma_s(t)$ is smooth, is one-to-one mapping, and has smooth inverse. Let Σ denote the two-dimensional submanifold spanned by the curves $\gamma_s(t)$. We may chose s and t as coordinates of Σ. The vector field $T^a = (\partial/\partial t)^a$ is tangent to the family of geodesics and thus satisfies

$$T^a \nabla_a T^b = 0.$$



A one-parameter family of geodesics γ_s, with tangent T^a and deviation X^a.

Definition

The vector field X^a represents the displacement to an infinitesimally nearby geodesic, and is called the *deviation vector*.

There is "*gauge freedom*" in X^a in a sense that X^a changes by addition of a multiple of T^a under change of affine parametrization of the geodesics $\gamma_s(t), t \to t' = b(s)t + c(s)$.

It is worth noting that in the case where the geodesic arise from the derivative operator (associated with the metric g_{ab}), X^a always can be chosen orthogonal to T^a.

Automatically constant along each geodesic
↓

Proof:

1.) By re-scaling t by an s-independent factor, we may ensure that $g_{ab}T^aT^b$ ***does not vary*** with s.

Since X^a and T^a are coordinate vector fields, ***they commute***:

$$T^b\nabla_b X^a = X^b\nabla_b T^a;$$

so by the same calculation as on slide 9, we see that $\boldsymbol{X^aT_a}$ ***is constant along each geodesic***.

2.) By further reparametrizing each $\gamma_s(t)$ by adding a constant (depending on s) to t, we may ensure that the curve $C(s)$ comprising the points with $t = 0$ is ***orthogonal to all the geodesics***.

→ Thus, with this affine parametrization of $\gamma_s(t)$, we have $X_aT^a = 0$ at $t = 0$ and hence $X^aT_a = 0$ everywhere.

commuting term on the last slide

1.) The quantity $v^a = T^b \nabla_b X^a$ gives the *rate of change* along a geodesic of the displacement to an infinitesimally nearby geodesic.

→ Thus, we may interpret v^a as the "*relative velocity*" of an infinitesimally nearby geodesic. Similarly, we may interpret

$$a^a = T^c \nabla_c v^a = T^c \nabla_c (T^b \nabla_b X^a)$$

as the "*relative acceleration*" of an infinitesimally nearby geodesic.

commuting term on the last slide

2.) We now shall derive an equation which relates a to the Riemann tensor. We have

$$\begin{aligned} a^a &= T^c \nabla_c \left(T^b \nabla_b X^a\right) = T^c \nabla_c \left(X^b \nabla_b T^a\right) \\ &= (T^c \nabla_c X^b)(\nabla_b T^a) + X^b T^c \nabla_c \nabla_b T^a \\ &= (X^c \nabla_c T^b)(\nabla_b T^a) + X^b T^c \nabla_c \nabla_b T^a - R_{cbd}{}^a X^b T^c T^d \\ &= X^c \nabla_c \left(T^b \nabla_b X^a\right) - R_{cbd}{}^a X^b T^c T^d = -R_{cbd}{}^a X^b T^c T^d. \end{aligned}$$

← expansion of bracets

← *Geodetic deviation equation*

The geodesic deviation equation yields the final characterization of curvature we sought:

We have $a^a = 0$ for all families of geodesics if and only if $R_{abc}{}^d = 0$. Thus some geodesics will accelerate towards or away from each other if and only if $R_{abc}{}^d \neq 0$.

Thank you for your attention!

Lecture notes on the course and other materials (e.g. recommended text books) will be available shortly at
https://wigner.hu/~barta/GRcourse2020/

Dániel Barta
E-mail: barta.daniel@wigner.hu