Kovariant derivative:

Def:
A derivative op., ∇, on a manifold M is a map which takes each C^∞ tensor field of type (k,l) to a C^∞ tensorfield of type $(k,l+1)$

$$\nabla : \mathcal{T}(k,l) \longrightarrow \mathcal{T}(k,l+1)$$

and satisfies the following 5 properties:

i) Linearity: $\forall\ A, B \in \mathcal{T}(k,l)$ and $\alpha, \beta \in \mathbb{R}$

$$\nabla_c \left(\alpha A^{a_1 \dots a_k}{}_{b_1 \dots b_l} + \beta B^{a_1 \dots a_k}{}_{b_1 \dots b_l}\right) = \alpha \nabla_c A^{a_1 \dots a_k}{}_{b_1 \dots b_l} + \beta \nabla_c B^{a_1 \dots a_k}{}_{b_1 \dots b_l}$$

ii) Leibnitz rule: $\forall A \in \mathcal{T}(k,l)$, $B \in \mathcal{T}(k',l')$

$$\nabla_e \left[A^{a_1 \dots a_k}{}_{b_1 \dots b_l} B^{c_1 \dots c_{k'}}{}_{d_1 \dots d_{l'}}\right] = \left[\nabla_e A^{a_1 \dots a_k}{}_{b_1 \dots b_l}\right] B^{c_1 \dots c_{k'}}{}_{d_1 \dots d_{l'}} + A^{a_1 \dots a_k}{}_{b_1 \dots b_l} \left[\nabla_e B^{c_1 \dots c_{k'}}{}_{d_1 \dots d_{l'}}\right]$$

iii) Commutativity with Contraction: $\forall\ A \in \mathcal{T}(k,l)$

$$\nabla_d \left(A^{a_1 \dots c \dots a_k}{}_{b_1 \dots c \dots b_l}\right) = \nabla_d A^{a_1 \dots c \dots a_k}{}_{b_1 \dots c \dots b_l}$$

iv) Consistency with the notion of tangent vec. as directional derivatives on scalar fields: $\forall\ f \in \mathcal{F}$ and $\forall\ t^a \in V_p$
$t(f) = t^a \nabla_a f$

v) Torsion free: $\forall\ f \in \mathcal{F}$
$\nabla_a \nabla_b f = \nabla_b \nabla_a f$

From condition 5 and 4 with the Leibnitz allow us to derive a simple expression for the comutator of two vector fields v^a, w^b in terms of any deri. op. ∇_a. Applied to any C^∞ func. f

$$[v,w](f) = v\{w(f)\} - w\{v(f)\} = v^a \nabla_a (w^b \nabla_b f) - w^a \nabla_a (v^b \nabla_b f)$$
$$= \{v^a \nabla_a w^b - w^a \nabla_a v^b\} \nabla_b f$$

$$[v,w]^b = v^a \nabla_a w^b - w^a \nabla_a v^b$$

Lets show that the derivative op. exist:

$!\ \psi$ be a coord. system and $\{\partial/\partial x^\mu\}$ and $\{dx^\mu\}$ be the associated coord. bases. Then in the region covered by these coords. we may define a derivative op. ∂_a, called an ordinary derivative,

$\forall\ C^\infty$ tensor field $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$ $\longrightarrow$ we take its comp. $T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$ in this coord. basis and define

$$\partial_c T^{a_1 \dots a_k}{}_{b_1 \dots b_l} \longrightarrow \frac{\partial T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}}{\partial x^\sigma}$$

All 5 cond. follow immediately from the standard prop. of partial derivatives.
The equality of mixed partial derivatives, the 5th cond. holds for all tensor field, not just for scalar fields.

$\Rightarrow$ given a coord. sys. ψ, we can construct an associated der. op. ∂_a.

How unique are der. op?

By cond 4, any 2 der. op. ∇_a and $\tilde{\nabla}_a$ must agree in their action on scalar fields. To investigate their disagreement on tensors of next highest rank. Let ω_b be a dual vec. field and consider the difference $\tilde{\nabla}_a(f\omega_b) - \nabla_a(f\omega_b)$ for arbitrary scalar field f. By the Leibnitz rule:

$$\tilde{\nabla}_a(f\omega_b) - \nabla_a(f\omega_b) = (\tilde{\nabla}_a f)\omega_b + f\tilde{\nabla}_a\omega_b - (\nabla_a f)\omega_b - f\nabla_a\omega_b = f(\tilde{\nabla}_a\omega_b - \nabla_a\omega_b)$$

at point p, $\tilde{\nabla}_a\omega_b$ and $\nabla_a\omega_b$ each depend on how ω_b changes as one moves away from p. $\rightarrow$ $\tilde{\nabla}_a\omega_b - \nabla_a\omega_b$ depend only on the value of ω_b at p. $\rightarrow$ $\omega_b' = \omega_b$ at p.
$\rightarrow$ $\omega_b \Leftrightarrow \omega_b'$.

$\omega_b' - \omega_b$ vanishes at p we can find C^∞ func. $f_{(\alpha)}$, which vanish at p and C^∞ dual vect. field $\mu_b^{(\alpha)}$

$$\omega_b' - \omega_b = \sum_{\alpha=1}^{n} f_{(\alpha)}\mu_b^{(\alpha)}$$

$$\tilde{\nabla}_a(\omega_b' - \omega_b) - \nabla_a(\omega_b' - \omega_b) = \sum_\alpha \{\tilde{\nabla}_a(f_{(\alpha)}\mu_b^{(\alpha)}) - \nabla_a(f_{(\alpha)}\mu_b^{(\alpha)})\}$$

$$= \sum_\alpha f_{(\alpha)}\{\tilde{\nabla}_a\mu_b^{(\alpha)} - \nabla_a\mu_b^{(\alpha)}\} = 0 \qquad / f_{(\alpha)} = 0$$

$$\tilde{\nabla}_a\omega_b' - \nabla_a\omega_b' = \tilde{\nabla}_a\omega_b - \nabla_a\omega_b \quad \square$$

Thus we have shown that $\tilde{\nabla}_a - \nabla_a$ defines a map of dual vec. at p to tensors of type $(0,2)$ at p.

By prop 1, this map is linear. Consequently $(\tilde{\nabla}_a - \nabla_a)$ defines a tensor of type $(1,2)$ at p, $C^c{}_{ab}$

$$\nabla_a\omega_b = \tilde{\nabla}_a\omega_b - C^c{}_{ab} \quad (*)$$

This displays the possible disagreement of the actions of $\tilde{\nabla}_a$ and ∇_a on dual vec. fields.

A symmetry prop. of $C^c{}_{ab}$ follows from cond. 5. If we let $\omega_b = \nabla_b f = \tilde{\nabla}_b f$:

$$\nabla_a\nabla_b f = \tilde{\nabla}_a\tilde{\nabla}_b f - C^c{}_{ab}\nabla_c f$$

$\nabla_a\nabla_b f$ and $\tilde{\nabla}_a\tilde{\nabla}_b f$ symmetric in a,b $\rightarrow$ $C^c{}_{ab}$ also has this prop. $\rightarrow$ $C^c{}_{ab} = C^c{}_{ba}$
It need not hold if cond 5 drops.

The difference in action must hold on vec. field and all higher rank tensor fields is determined by eq $(*)$, the Leibnitz rule, and prop. 4.
$\forall\, t^a$ and ω_b, prop (4) tells us:
vec   one-form

$$(\tilde{\nabla}_a - \nabla_a)(\omega_b t^b) = 0$$

by the Leibnitz rule:

$$(\tilde{\nabla}_a - \nabla_a)(\omega_b t^b) = (C^c{}_{ab}\omega_c)t^b + \omega_b(\tilde{\nabla}_a - \nabla_a)t^b$$

index substituting on contracted indices

$$\omega_b[(\tilde{\nabla}_a - \nabla_a)t^b + C^b{}_{ac}t^c] = 0 \qquad \forall \omega_b$$

$$\nabla_a t^b = \tilde{\nabla}_a t^b + C^b{}_{ac} t^c$$

We can derive now the general formula for the action of ∇_a on an arbitrary tensor field in terms of $\tilde{\nabla}_a$ and $C^c{}_{ab}$. For $T \in \mathcal{T}(k,l)$

$$\nabla_a T^{b_1\dots b_k}{}_{c_1\dots c_l} = \tilde{\nabla}_a T^{b_1\dots b_k}{}_{c_1\dots c_l} + \sum_i C^{b_i}{}_{ad} T^{b_1\dots d\dots b_k}{}_{c_1\dots c_l} - \sum_j C^d{}_{ac_j} T^{b_1\dots b_k}{}_{c_1\dots d\dots c_l}$$

The most important application of above eq. arises from the case where $\tilde{\nabla}_a$ is an ordinary derivative op. ∂_a. In the case, the tensor field $C^c{}_{ab}$ is denoted $\Gamma^c{}_{ab}$ and called a Cristoffel symbol.

$$\nabla_a t^b = \partial_a t^b + \Gamma^b{}_{ac} t^c$$

Given a derivative op. ∇_a we can define the notion of parallel transporte of a vec. along a curve C with tangent t^a. A vector v^a given at each point on the curve is said to be parallelly transported as one moves along the curve if the eq

$$t^a \nabla_a v^b = 0$$

is satisfied along the curve.

$$\boxed{t^a \nabla_a T^{b_1\dots b_k}{}_{c_1\dots c_l} = 0} \quad \text{More generally}$$

$$t^a \partial_a v^b + t^a \Gamma^b{}_{ac} v^c = 0$$

in terms of the comp. in the coord. basis and the parameter t along the curve

$$\boxed{\frac{dv^\nu}{dt} + \sum_{\mu,\lambda} t^\mu \Gamma^\nu{}_{\mu\lambda} v^\lambda = 0} \quad (\Delta)$$

- parallel transport of v^a depends only on the values of v^a on the curve
- prop. ODE $\Longrightarrow$ (Δ) eq always has unique solution $\forall$ initial value of v_a
- a vec. at point p uniquely defines a "parallel transported vec" everywhere else on the curve.
- use this notion of parallel transport to identify (i.e, map into each other) the tangent sp. V_p & V_q of point p & q if we are given a derivative op. and a curve connecting p & q.
- It is called a connection.

We show now that if one is given a metric g_{ab} on the manifold, a natural choice of derivative op is uniquely picked out.

$g_{ab} \rightarrow$ natural condition $\rightarrow$ parallel trans.

Given v^a & w^a $\rightarrow$ inner prod. $g_{ab} v^a w^b$ remain unchanged if we para. trans. them along the curve.

$$t^a \nabla_a (g_{bc} v^b w^c) = 0 \qquad / \text{Leibnitz rule}$$

$$t^a v^b w^c \nabla_a g_{bc} = 0 \xLeftarrow{\text{iff}} \nabla_a g_{bc} = 0$$

Theorem: ! g_{ab} be a metric. Then there exists a unique der. op. ∇_a satisfying $\nabla_a g_{bc} = 0$.

Proof:

! $\tilde{\nabla}_a$ be any der. op., e.g., an ord. der. op. associated with the coord. sys. We attempt to solve for $C^c{}_{ab}$ so that the der. op. determined by $\tilde{\nabla}_a$ and $C^c{}_{ab}$ will satisfy the req. prop. We will prove the theorem by showing that a unique solution for $C^c{}_{ab}$ $\exists$.

$$0 = \nabla_a g_{bc} = \tilde{\nabla}_a g_{bc} - C^d{}_{ab} g_{dc} - C^d{}_{ac} g_{bd}$$

$$\underbrace{C_{cab} + C_{bac} = \tilde{\nabla}_a g_{bc} \qquad C_{cba} + C_{abc} = \tilde{\nabla}_b g_{ac}}_{① \ \Sigma} \qquad \underbrace{C_{bca} + C_{acb} = \tilde{\nabla}_c g_{ab}}_{-①}$$

Using the sym. prop. $C^c{}_{ab}$

$$2C_{cab} = \tilde{\nabla}_a g_{bc} + \tilde{\nabla}_b g_{ac} - \tilde{\nabla}_c g_{ab}$$

$$C^c{}_{ab} = \frac{1}{2} g^{cd} \{ \tilde{\nabla}_a g_{bd} + \tilde{\nabla}_b g_{ad} - \tilde{\nabla}_d g_{ab} \}$$

This choce of $C^c{}_{ab}$ solves $\nabla_a g_{bc} = 0$ and is manifestly unique. □

In terms of an ordinary der. op. the cristofel symbol is

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd} \{ \partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \}$$

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} \sum_\sigma g^{\rho\sigma} \left\{ \frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right\} \qquad \text{coord basis comp.}$$

The transformation law for Cristoffel symbols:

$$\Gamma'^c{}_{ab} = \frac{\partial x'^c}{\partial x^c} \left(\frac{\partial^2 x^c}{\partial x'^a \partial x'^b} + \frac{\partial x^a}{\partial x'^a} \frac{\partial x^b}{\partial x'^b} \Gamma^c{}_{ab} \right)$$

the $\Gamma^c{}_{ab}$ do not transform as the comp. of a tensor