INTRODUCTION TO GENERAL RELATIVITY

Tensor calculus on manifolds

TENSOR FIELDS ON MANIFOLDS

- Last we considered vector (and dual vector) space. Now, let's turn our attention to *tensor fields*.
- It is clear that we can generalizethe notion of a vector: in the tangent space T_p, we can define a *tensor T* as a linear map from some number of vectors to the real numbers.

The *rank of the tensor* is the number of vectors it has for its arguments. For example, we can write a third-rank tensor as $T(\cdot,\ \cdot,\ \cdot)$. Once again, we denote the number that the tensor T produces from the vectors $\mathrm{u}, v, w \in \boldsymbol{V}$ by

$$T(\mathrm{u}, v, w).$$

Clearly, from our above discussion, any vector is a rank-1 tensor. → Higher-rank tensors thus constitute a generalization of the concept of a vector. For example, a particularly important second-rank tensor is the *metric tensor:* $\boxed{g(\mathrm{u}, v) = u \cdot v}$.

Defined as a linear map of two vectors into the number; that is their inner product

Definition

Let V a finite dimensional vector space and let $\boldsymbol{V}^*$ denote its dual vector space. A *tensor* T, of type (k,l) over $\boldsymbol{V}$ is a *multilinear map*

$$T := \underbrace{\boldsymbol{V}^* \times \ldots \times \boldsymbol{V}^*}_{k} \times \underbrace{\boldsymbol{V} \times \ldots \times \boldsymbol{V}}_{l} \to \mathbb{R}.$$

The fact that a tensor is a *linear* map of the vectors into the reals is particularly useful. For simplicity, let us consider a rank-1 tensor. Linearity means that, for general vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and general scalars α and β,

$$t(\alpha \boldsymbol{u} + \beta \boldsymbol{v}) = \alpha t(\boldsymbol{u}) + \beta t(\boldsymbol{v}).$$

Similar expansions may be performed for tensors of higher rank. For a second-rank tensor, for example, we can write

$$t(\alpha \boldsymbol{u} + \beta \boldsymbol{v}, \gamma \boldsymbol{w} + \epsilon \boldsymbol{z}) = \alpha t(\boldsymbol{u}, \gamma \boldsymbol{w} + \epsilon \boldsymbol{z}) + \beta t(\boldsymbol{v}, \gamma \boldsymbol{w} + \epsilon \boldsymbol{z})$$
$$= \alpha\gamma t(\boldsymbol{u}, \boldsymbol{w}) + \alpha\epsilon t(\boldsymbol{u}, \boldsymbol{z}) + \beta\gamma t(\boldsymbol{v}, \boldsymbol{w}) + \beta\epsilon t(\boldsymbol{v}, \boldsymbol{z}).$$

Because of the multilinearity property, a tensor is uniquely specified by giving its values on vectors in a basis $\{v_\mu\}$ of $\boldsymbol{V}$ and its dual basis $\{v^{\nu *}\}$ of $\boldsymbol{V}^*$.

Since there are n^{k+1} independent ways of filling the slots of a tensor of (k,l) with such basis vectors (where $n = \dim \boldsymbol{V} = \dim \boldsymbol{V}^*$), the dimension of the vector space $\mathcal{T}(k,l)$ is n^{k+1}

REMARKS

- According to the above definition, a tensor of type $(0,1)$ is precisely a dual vector.
- Similarly, a tensor of type $(1,0)$ is an element of $\boldsymbol{V}^{**}$.

However, since we identify $\boldsymbol{V}^{**}$ with $\boldsymbol{V}$, we may view tensors of higher type in many different ways.

For example: A tensor T of type $(1,1)$ is multilinear map from $\boldsymbol{V}^* \times \boldsymbol{V}$ into $\mathbb{R}$. Hence, if $v \in \boldsymbol{V}$, $T(\cdot, v)$ is an element of $\boldsymbol{V}^{**}$, which we identify with an element of $\boldsymbol{V}$. Thus given a vector in $\boldsymbol{V}$, T produces another vector in $\boldsymbol{V}$ in a linear fashion.

ELEMENTARY OPERATIONS WITH TENSORS

With the rules for ***adding*** and ***scalar multiplying maps***, the collection $\mathcal{T}(k,l)$ of all tensors of type (k,l) has the structure of a vector space.

Addition (and subtraction)

It is clear from the definition of a tensor that the sum and difference of two tensors of rank N are both themselves tensors of rank N. For example, the covariant components (say) of the sum $\boldsymbol{s}$ and difference $\boldsymbol{d}$ of two rank-2 tensors are given straightforwardly by

$$s_{ab} = \boldsymbol{s}(\boldsymbol{e}_a, \boldsymbol{e}_b) = \boldsymbol{t}(\boldsymbol{e}_a, \boldsymbol{e}_b) + \boldsymbol{r}(\boldsymbol{e}_a, \boldsymbol{e}_b) = t_{ab} + r_{ab},$$

$$d_{ab} = \boldsymbol{d}(\boldsymbol{e}_a, \boldsymbol{e}_b) = \boldsymbol{t}(\boldsymbol{e}_a, \boldsymbol{e}_b) - \boldsymbol{r}(\boldsymbol{e}_a, \boldsymbol{e}_b) = t_{ab} - r_{ab}.$$

Multiplication by a scalar

If $\boldsymbol{t}$ is a rank-N tensor then so too is $\alpha\boldsymbol{t}$, where is some arbitrary real constant. Clearly, its components are all multiplied by α.

Tensor product (outer product)

The *outer or tensor product* of two tensors produces a tensor of *higher rank*.

Definition

Let T and S be two tensor at x of type (k, l) and (p, q). Then the *tensor product* or *outer product* $T \otimes S$ is the tensor at x of type $(k + p, l + q)$ defined by

$$T \otimes S\left(v_1, \dots, v_{k+p}, \omega_1, \dots, \omega_{l+q}\right) = T(v_1, \dots, v_k, \omega_1, \dots, \omega_l) \times S(v_{k+1}, \dots, v_{k+p}, \omega_{l+1}, \dots, \omega_{l+q}).$$

for all vectors $v_1, \dots, v_{k+p} \in T_x M$ and all covectors $\omega_1, \dots, \omega_{l+q} \in T_x^* M$.

Contraction (and inner product)

The *contraction of a tensor* is performed by summing over the basis and dual basis vectors in two of its vector arguments, and it results in a tensor of *lower rank*.

Definition

Let T be a tensor of type (k, l) at x, with k and l at least 1. Then T has components ${T_{i_1,\dots,i_k}}^{j_1,\dots,j_l}$ as before. Then there is a tensor of type $(k-1, l-1)$ which has components

$$\sum_{a=1}^{n} {T_{i_1,\dots,i_{k-1}a}}^{j_1,\dots,j_{l-1}a} .$$

This tensor is called a *contraction* of T. (If k and l are large then there will be many such contractions, depending on which indices we choose to sum over).

A special case is where T is a tensor of type $(1,1)$. This takes a vector and gives another vector, and so is nothing other than ***a linear operator on*** TM. There is a unique contraction in this case, which is just the ***trace of the operator***:

$$\operatorname{tr} A = \sum_{i=1}^{n} {A_i}^i .$$

COMPONENTS OF TENSORS

When a tensor is evaluated with combinations of basis and dual basis vectors it yields its ***components*** in that particular basis. For example, the covariant and contravariant components of the rank-1 tensor (vector) in the basis $\boldsymbol{e}_a$ are given by

$$\boldsymbol{t}(\boldsymbol{e}_a) = t_a \qquad \text{and} \qquad \boldsymbol{t}(\boldsymbol{e}^a) = t^a.$$

Consider now a second-rank tensor $\boldsymbol{t}(\cdot,\ \cdot)$. Its ***covariant*** and ***contravariant*** components are given by

$$\boldsymbol{t}(\boldsymbol{e}_a, \boldsymbol{e}_b) = t_{ab} \qquad \text{and} \qquad \boldsymbol{t}(\boldsymbol{e}^a, \boldsymbol{e}^b) = t^{ab}.$$

For tensors of rank 2 and higher, however, we can also define sets of ***mixed*** components. For a rank-2 tensor there are two possible sets of mixed components,

$$\boldsymbol{t}(\boldsymbol{e}^a, \boldsymbol{e}_b) = t^a{}_b \qquad \text{and} \qquad \boldsymbol{t}(\boldsymbol{e}_a, \boldsymbol{e}^b) = t_a{}^b.$$

SYMMETRIES OF TENSORS

A second-rank tensor t is called ***symmetric*** or ***antisymmetric*** if, for ***all*** pairs of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$,

$$\boxed{\boldsymbol{t}(\boldsymbol{u},\boldsymbol{v}) = \pm\boldsymbol{t}(\boldsymbol{v},\boldsymbol{u})},$$

with the plus sign for a symmetric tensor and the minus sign for an antisymmetric tensor. Setting $\boldsymbol{u} = \boldsymbol{e}_a$ and $\boldsymbol{v} = \boldsymbol{e}_b$, we see that the covariant components of a symmetric or antisymmetric tensor satisfy

$$t_{ab} = \pm t_{ba}.$$

By using different combinations of basis and dual basis vectors we also see that, for such a tensor,

$$t^{ab} = \pm t^{ba} \quad \text{and} \quad t_a{}^b = \pm t^b{}_a.$$

An arbitrary rank-2 tensor can always be split uniquely into the sum of its symmetric and antisymmetric parts. For illustration let us work with the covariant components tab of the tensor in some basis. We can always write

$$t_{ab} = \frac{1}{2}(t_{ab} + t_{ba}) + \frac{1}{2}(t_{ab} - t_{ba}),$$

which is clearly the sum of a symmetric and an antisymmetric part. A notation frequently used to denote the components of the symmetric and antisymmetric parts is

$$\boxed{t_{(ab)} = \frac{1}{2}(t_{ab} + t_{ba})} \quad \text{and} \quad \boxed{t_{[ab]} = \frac{1}{2}(t_{ab} - t_{ba})}.$$

Generally:

$$t_{(ab\dots c)} = \frac{1}{N!}(\text{sum over all permutations of the indices } a, b, \dots, c),$$

$$t_{[ab\dots c]} = \frac{1}{N!}(\text{alternating sum over all permutations of the indices } a, b, \dots, c).$$

We may extend the notation still further in order to define tensors that are symmetric or antisymmetric to permutations of particular subsets of their indices. To illustrate this, let us consider the covariant components t_{abcd} of a fourth-rank tensor. Typical expressions might include:

$$t_{(ab)cd} = \tfrac{1}{2}(t_{abcd} + t_{bacd}),$$

$$t_{a[b|c|d]} = \tfrac{1}{2}(t_{abcd} - t_{adcb}),$$

$$t_{(a|b|cd)} = \tfrac{1}{6}(t_{abcd} + t_{abdc} + t_{dbac} + t_{dbca} + t_{cbda} + t_{cbad}),$$

$$t_{[ab](cd)} = \tfrac{1}{2}\left[t_{ab(cd)} - t_{ba(cd)}\right] = \tfrac{1}{2}\left[\tfrac{1}{2}(t_{abcd} + t_{abdc}) - \tfrac{1}{2}(t_{bacd} + t_{badc})\right]$$
$$= \tfrac{1}{4}(t_{abcd} + t_{abdc} - t_{bacd} - t_{badc}).$$

The symbols are used to exclude unwanted indices from the (anti-) symmetrisation implied by ( ) and [ ].

THE METRIC TENSOR

The most important tensor that one can define on a manifold is the *metric tensor* g. Intuitively, a metric is supposed to tell us the ”*infinitesimal squared distance*” (ds^2) associated with ”*infinitesimal displacement*” (dx). *(Recall that last week's lecture we have seen that the notion of an infinitesimal displacement is precisely captured by the concept of a tangent vector.)*

Requirements for the metric tensor:

1. Since the ”*infinitesimal squared distance*” should be *quadratic in the displacement* , a metric tensor g should be a multilinear map from $\boldsymbol{V}_p \times \boldsymbol{V}_p \to \mathbb{R}$, i.e. a of type (0,2).
2. The metric tensor is also required to be *symmetric*: $g(v_1, v_2) = g(v_2, v_3)$ for all $v_1, v_2 \in \boldsymbol{V}_p$
3. And *non-degenerate* as well: $g(v, v_1) = 0$ for all $v \in \boldsymbol{V}_p$ in case $v_1 = 0$.

In a coordinate basis, we may expand a metric g in terms of its components g_{ab} as

$$\boxed{g = \sum_{a,b} g_{ab} dx^a \otimes dx^b} .$$

Sometimes the notation ds^2 for the *infinitesimal displacement* is used in place of g to represent the metric tensor, in which case we write the above as

$$\boxed{ds^2 = g_{ab} dx^a dx^b} .$$

We have ommited writting the ***outer product sign*** *between dx^a and dx^b*

We have ommited the ***summation sign*** *for twice occuring indices*

Given a metric g, we always find an *orthonormal basis* $\{v_1, \ldots, v_n\}$ of tangent space T_p at each point $p \in \mathcal{M}$, i.e., a basis such that $g(v_a, v_b) = 0$ if $a \neq b$ and $g(v_a, v_b) = \pm 1$.

- There are many other bases at p, but the number of basis vectors with $g(v_a, v_b) = +1$ and $g(v_a, v_b) = -1$ are independent of choice of orthonormal basis.
- The number of + and – signs occuring is called the *signature* of the metric.
  - Positive definite metrics are called *Riemannian*;
  - Mostly positive metrics are called *pseudo-Riemannian*
  - Metrics on spacetimes (one minus and the rest plus) are called *Lorentzian*.

In ordinary diferential geometry, one usually deals with positive definite metrics, i.e., metrics with signature $(-,+,+,+)$

Its covariant and contravariant components are given by

$$\boxed{g_{ab} = g(v_a, v_b) = v_a \cdot v_b} \quad \text{and} \quad \boxed{g^{ab} = g\left(v^a, v^b\right) = v^a \cdot v^b}.$$

The matrix $[g^{ab}]$ containing the contravariant components of the metric tensor is the inverse of the matrix $[g_{ab}]$ that contains its covariant components. The mixed components of $\boldsymbol{g}$ are given by

$$\boxed{g\left(v^b, v_a\right) = g\left(v_a, v^b\right) = \delta_a^b},$$

where the last equality is a result of the reciprocity relation between basis vectors and their duals.

RAISING AND LOWERING TENSOR INDICES

The contravariant and covariant components of the metric tensor can be used for raising and lowering general tensor indices, just as they are used for vector indices. As we have seen, when a tensor acts on different combinations of basis and dual basis vectors it yields different components. Consider, for example, a third-rank tensor $\boldsymbol{t}$. Its covariant components are given by

$$\boldsymbol{t}(\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}_c) = t_{abc}$$

whereas one possible set of *mixed* components of the tensor is given by

$$\boldsymbol{t}(\boldsymbol{e}_a, \boldsymbol{e}_b, \boldsymbol{e}^c) = t_{ab}{}^{c}.$$

As we stated earlier, in general these two sets of components will differ, since the basis and dual basis vectors are related by the metric: $\boldsymbol{e}_c = g_{cd}\boldsymbol{e}^d$. Thus, for example,

$$t_{abc} = g_{cd} t_{ab}{}^{d}$$

In a similar way we can raise or lower more than one index at a time. For example

$$t^{a}{}_{bc} = g^{ad} g_{ce} t_{db}{}^{e}.$$

TENSORS AND COORDINATE TRANSFORMATIONS

The description of tensors as a geometrical objects lends itself naturally to a discussion of the behaviour of tensor components under a coordinate transformation $x^a \longrightarrow x'^a$ on the manifold.

There is a simple relationship between the coordinate basis vectors $\boldsymbol{e}_a$ associated with the coordinate system x^a and the coordinate basis vectors $\boldsymbol{e}'_a$ a associated with a new system of coordinates x'^a: at any point P the two sets of coordinate basis vectors are related by

$$\boldsymbol{e}'_a = \frac{\partial x^b}{\partial x'^a} \boldsymbol{e}_b,$$

where the partial derivative is evaluated at the point P. A similarrelationship holds between the two sets of dual basis vectors:

$$\boldsymbol{e}'^a = \frac{\partial x^b}{\partial x'^a} \boldsymbol{e}^b.$$

→ *We can now calculate how the components of any general tensor must transform under the coordinate transformation!*

The contravariant components of a vector $\boldsymbol{t}$ in the new coordinate basis are given by

$$t'^a = \boldsymbol{t}(\boldsymbol{e}'^a) = \frac{\partial x'^a}{\partial x^b} \boldsymbol{t}\left(\boldsymbol{e}^b\right) = \frac{\partial x'^a}{\partial x^b} t^b.$$

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Thank you for your attention!

Lecture notes on the course and other materials (e.g. recommended text books) will be available shortly at
https://wigner.hu/~barta/GRcourse2020/

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