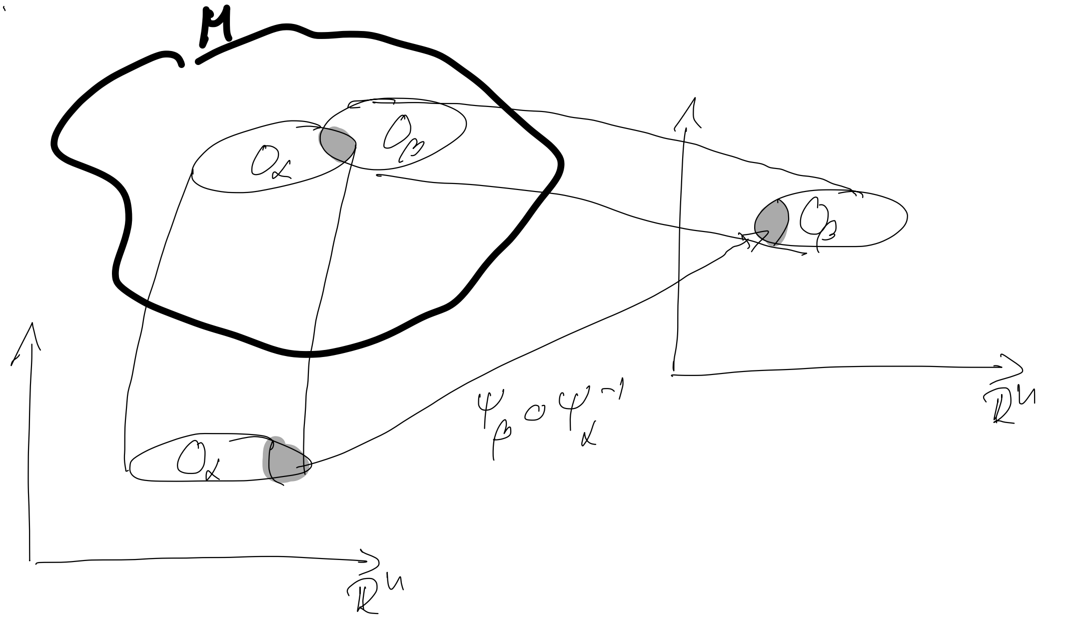
Def: A function $f: \mathbb{R}^n \to \mathbb{R}^n$ is said to be class $C^r$, if for any of its var. it can be continuously partially differentiated $r$-times. This way the class of continuous functions is $C^0$, while those functions that have any order of partial derivative is class $C^\infty$ or called smooth. If the function also same as its Taylor sequence, then we call it analitical or class $C^\omega$.

Def: The $(M, \mathcal{T})$ top. sp. is class $C^r$, n-dim, real differentiable manifold, if there for it $\exists$ subsets of pairs of points $\{(O_\alpha, \psi_\alpha)\}$ such that they satisfying the following properties:

(i) The seq. of sets $\{O_\alpha\}$ is open cover of $M$, i.e. $M \subset \bigcup_\alpha O_\alpha$.

(ii) $\forall \alpha$, there is one-to-one, onto, map $\psi_\alpha: O_\alpha \to U_\alpha$, where is $U_\alpha$ is an open subset of $\mathbb{R}^n$.

(iii) If any two set $O_\alpha$ & $O_\beta$ over lap, $O_\alpha \cap O_\beta \neq \emptyset$, we can consider the map $\psi_\beta \circ \psi_\alpha^{-1}: \psi_\alpha[O_\alpha \cap O_\beta] \to \psi_\beta[O_\alpha \cap O_\beta]$. We require these subsets of $\mathbb{R}^n$ to be class $C^\infty$.




Matematicians:

$(O_\alpha, \psi_\alpha)$ chart $\longrightarrow$ $\{(O_\alpha, \psi_\alpha)\}$ atlas

Physicists:

- $(O_\alpha, \psi_\alpha)$ local coord. sytem
- $\psi_\beta \circ \psi_\alpha^{-1}$ coord. transzformation
- $!\ \{(O_\alpha, \psi_\alpha)\}$ maximal

In chase of some $(O, \psi)$ in stead of using $\psi$ many times we use the coord. $(x_1, \dots, x_n)$ expressed by it.

Examples:

1.) Euclidian sp., $\mathbb{R}^n$ $(\mathbb{E}^n)$, provides a trivial example of manifolds, which can be covered by a single chart ($O = \mathbb{R}^n$, $\psi$ = identity map)

2.) 2-sphere $S^2$:

$$S^2 = \{(x^1, x^2, x^3) \subset S^2 \mid \pm x^i > 0\}$$

The entire $S^2$ cannot be mapped into $\mathbb{R}^2$ in a contiuous, one-to-one manner, but "pieces" of $S^2$ can, and these can be "smoothly sewn together. For example, if we define the 6 hemispherical open sets $O_i^\pm$ for $i = 1, 2, 3$ by

$$O_i^{\pm} = \{(x^1,x^2,x^3) \in S^2 \,|\, (x^1)^2+(x^2)^2+(x^3)^2 = 1\}$$

then $\{O_i^{\pm}\}$ covers $S^2$. Furthermore, each $O_i^{\pm}$ can be mapped homeomorphically into the open disk $D = \{(x,y) \in \mathbb{R}^2 \,|\, x^2+y^2 < 1\}$ in the plain via the "projections maps" $f_1^+ : O_1^+ \to D$, $f_1^- : O_1^- \to D$, etc., defined by $f_1^+(x^1,x^2,x^3) = (x^2,x^3)$, etc. The overlap func. $f_i^{\pm} \circ (f_j^{\pm})^{-1}$ can be checked to be $C^\infty$ in their domain of def. Thus, $S^2$ is a 2-dim manifold. In the similar manner, the n-dim sphere $S^n$ is seen to be a manifold.

In chase, if we want to express further structure on our manifold such as metric or integration interpreted on diff. manifolds, we assume that our manifold be paracompact

Def: An atlas $\{(O_\alpha, \psi_\alpha)\}$ is said to be locally finite if every point $p \in M$ has an open neighborhood which intersects only a finite number of the sets $O_\alpha$

Def: A connected Hausdorff manifolds is paracompact IFF it has a countable basis, i.e. there is a countable collection of open sets such that any open can be expressed as the union of members of this collection.

Def: Given two manifold $M$ and $M'$ of class $C^r$, a mapping $f: M \to M'$ is said to be differentiable of class $C^k$, $k \le r$, if for every chart $(O_\alpha, \psi_\alpha)$ of $M$ and every chart $(O'_\beta, \psi'_\beta)$ of $M'$ such that $f[\tilde{O}_\alpha \subset O_\alpha] \subset \tilde{O}'_\beta \subset O'_\beta$ the mapping $\psi'_\beta \circ f \circ \psi_\alpha^{-1}$ of $\psi_\alpha[\tilde{O}_\alpha]$ into $\psi'_\beta[\tilde{O}'_\beta]$ is differentiable of class $C^k$.

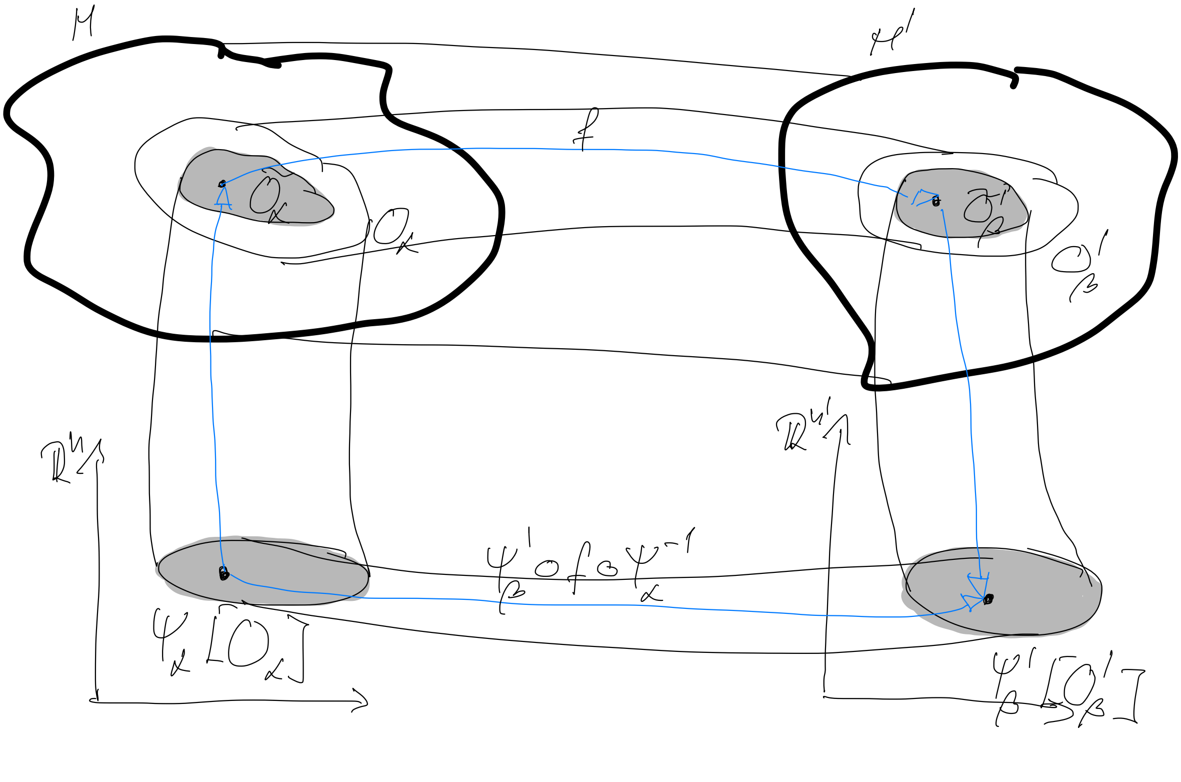
Examples:

1.) We choose $M$ to be any subset of $\mathbb{R}$. Then, by definition $f: M \subset \mathbb{R} \to M'$ expresses a class $C^k$ curve in $M'$.

2.) ! $M' = \mathbb{R}$. Then the above expression gives us the definition of class $C^k$ of func. $f: M \to \mathbb{R}$.

Def: ! $M$ and $M'$ be class $C^r$ manifolds. If $f: M \to M'$ is $C^r$, one-to-one, onto, and has $C^r$ inverse, $f$ is called deformorphic and $M$ and $M'$ are said to be diffeomorphic.

remark: Diffeomorphic manifolds have identical manifold structure.



Example:1

In the definition of deffeomorphism it is not enough, if we only require the $f: M \to M'$ to be one-to-one, onto, and it differetialibility. Because, for example the func. $f: \mathbb{R} \to \mathbb{R}$, where $f(x) = x^3$, is one-to-one, onto, and differentiable, but its inverse is not differentiable at point $f'(x) = 0$.