Definition: A topological space $(X, \mathcal{T})$ consist of a set X together with the collection $\mathcal{T}$ of subsets of X satisfying the following 3 condition:

(i) The union of an arbitrary collection of subsets, each of which is in $\mathcal{T}$: If $O_\alpha \in \mathcal{T}$ $\forall \alpha$, then $\bigcup_\alpha O_\alpha \in \mathcal{T}$.

(ii) The intersection of a finite number of sets in $\mathcal{T}$ is in $\mathcal{T}$: If $O_1, \dots, O_n \in \mathcal{T}$ then

$$\bigcap_{i=1}^{n} O_i \in \mathcal{T}.$$

(iii) The entire set X and the $\emptyset$ set are in $\mathcal{T}$

$\mathcal{T}$ is refered to as a topology on X, and subsets of X which are listed in the collection $\mathcal{T}$ are called open sets.

Examples:

1) $\mathcal{T} = \{$all subsets of $X\}$ – discret topology

2) $\mathcal{T} = \{X, \emptyset\}$ – indiscret topology

3) ! $X = \mathbb{R}$, the set of real numbers, & to defining $\mathcal{T}$ to consits of all subsets of $\mathbb{R}$ which can be expressed as unions of open intervals (a, b).

More generally, $\forall$ metric space, the collection of all subsets which can be expressed as unions of open balls yields a topology.

Def: If $(X, \mathcal{T})$ is topological space an A is any subsets of X, we may make A itself into a topological space by defining, $\mathcal{J}$, on A to consist of all subsets of A which can be expressed as intersection of elements of $\mathcal{T}$ with A, $\mathcal{J} = \{U \mid U = A \cap O,\ O \in \mathcal{T}\}$.

$\mathcal{J}$ is called the induced (or relative) topology.

Def: If $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ are topological spaces, we can make the product space $X_1 \times X_2 \equiv \{(x_1, x_2) \mid x_1 \in X_1,\ x_2 \in X_2\}$ into a topological space $(X_1 \times X_2, \mathcal{T})$ by defining $\mathcal{T}$ to consist of all subsets of $X_1 \times X_2$ which can be expressed as unions of sets of form $O_1 \times O_2$ with $O_1 \in \mathcal{T}_1$ & $O_2 \in \mathcal{T}_2$.

$\mathcal{T}$ is called a product topology.

Def: If $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are top. sp.-s, a map $f: X \to Y$ is said to be continuous, if the inverse image, $f^{-1}[O] = \{x \in X \mid f(x) \in O\}$, of every open set O in Y is an open set in X.

Def: If f is continuous, one-to-one, onto, and its inverse is continuous, f is called a homeomorphism and $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ are said to be homeomorphic.

Def: If $(X, \mathcal{T})$ is a top. sp., a $C \subset X$ is said to be closed if its complement $X - C \equiv \{x \in X / x \notin C\}$ is open.

Exm:
A closed interval $[a,b] \in \mathbb{R}$ (with the standard top. on $\mathbb{R}$) is a closed set.

Def: A topological space $(X, \mathcal{T})$ is said to be connected if the only subsets which which are both open and closed are the entire sp. X and $\emptyset$.

Def: If $(X, \mathcal{T})$ is top. sp. and $A \subset X$, the closure, $\bar{A}$, of A is defined as the intersection of all closed sets containing A.

Def: The interior, A°, of A defined as the union of all open sets contained within A.

Def: The boundary, denoted ∂A, consist of all points which lie in $\bar{A}$ but not in A°: $\partial A = \bar{A} \setminus A^{\circ}$.

Def: A top. sp. $(X, \mathcal{T})$ is said to be Hausdorff if $\forall$ distinct point $p, q \in X$, $p \neq q$, one can find open sets $O_p, O_q \in \mathcal{T}$ such that $p \in O_p$ & $q \in O_q$.

Def: If $(X, \mathcal{T})$ is top. sp. and $A \subset X$, a collection $\{O_\alpha\}$ of open sets is said to be an open cover of A if the union these sets contains A.
A sub collection of $\{O_\alpha\}$ which also covers A is refered to as a su bsubcover.

Def: The set A said to be compact if every open cover of A has a finite subcover.

Theorem:
A closed interval $[a,b]$ of real numbers is compact (with the standard top. on $\mathbb{R}$).

Theorem:
! $(X, \mathcal{T})$ be Hausdorff and ! $A \subset X$ be compact. $\Rightarrow$ A is closed.

Theorem:
! $(X, \mathcal{T})$ be compact and ! $A \subset X$ be closed. $\Rightarrow$ A is compact.

combining this 3 theorem:

Theorem:
A subset $A \subset \mathbb{R}$ is compact IF it is closed and bounded.

The property of compactness is easily proven to be preserved under continuous maps.

Theorem:
! $(X, \mathcal{T})$ and $(Y, \mathcal{J})$ be top. sp. Suppose $(X, \mathcal{T})$ is compact and $f : X \to Y$ is continuous. $\Rightarrow$ $f[X] \equiv \{y \in Y / y = f(x)\}$ is compact.

Theorem:
A continuous func. from a compact top. sp. into $\mathbb{R}$ is bounded and attains its maximum and minimum values.

The following theorem gives an imidict extension of results on compactness for $\mathbb{R}$ to results for $\mathbb{R}^n$

Theorem (Tychonoff):
! $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be compact top. sp. Then the product sp. $X_1 \times X_2$ is compact in the product topology.

Generalized to apply to the product of ∞ many top. sp.
A corollary of this and the above theorems:

Theorem:
A subset, A, of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Example:
- The n-dim sphere S^n in the induced top. is compact, since it is easily seen to be a closed and bounded subset of $\mathbb{R}^{n+1}$

Def: A sequence $\{x_n\}$ of points in a top. sp. $(X, \mathcal{T})$ is said to converge to point x if given any open neighborhood O of x there is an N such that $x_n \in O$ $\forall n > N$. The point x is said to be the limit of this sequence.

Def: A point $y \in X$ is said to be an accumulation point (or limit point) of $\{x_n\}$ if every open neighborhood of y contains ∞ many points of the sequence

However, in general top. sp. if y is accumulation point of $\{x_n\}$, it may not be even possible to find subsequence $\{y_n\}$ of points of sequence $\{x_n\}$ such that $\{y_n\}$ converge to y.
However, the extraction of subseq. convergent to y will always be possible:

Def: If $(X, \mathcal{T})$ is first countable, that is, if $\forall p \in X$ there is a countable collection $\{O_n\}$ of open sets such that every open neighborhood, O, of p at least contains one member of this collection

Def: Second countability: there is a countable collection of open sets such that every open set can be expressed as a union of sets in this collection.

Exm.:
- For $\mathbb{R}^n$, the open balls with rational radii centered on points with rational coordinates

Theorem (Bolzano-Weierstrass):
! $(X, \mathcal{T})$ be a top. sp. and $A \subset X$. If A is compact, then every seq. $\{x_n\}$ of points in A has an accumulation point lying in A. Conversely, if $(X, \mathcal{T})$ is 2nd countable and every seq. in A has an accumulation point in A, then A is compact. Thus in particular, if $(X, \mathcal{T})$ is 2nd countable, A is compact if and only if every seq. in A has a convergent subseq. whose limit lies in A.

Def: ! $(X, \mathcal{T})$ be top. sp. and $\{O_\alpha\}$ an open cover of X. An open cover $\{V_\beta\}$ is said to be a refinement of $\{O_\alpha\}$ if $\forall V_\beta$ there $\exists$ an O_α such that $V_\beta \subset O_\alpha$.

Def: The cover $\{V_\beta\}$ is said to be locally finite if each $x \in X$ has an open neighborhood W such that only finitely many V_β satisfy $W \cap V_\beta \neq \emptyset$.

Def: The top. sp. $(X, \mathcal{T})$ is said to be paracompact if every open cover $\{O_\alpha\}$ of X has a localy finite refinement $\{V_\beta\}$.