INTRODUCTION TO GENERAL RELATIVITY

Curvature

RIEMANN CURVATURE TENSOR

- During the last lesson we had seen that given a derivative operator, there exists a notion of how to parallel transport a vector from p to q along a curve C.
- We can use the path dependence of parallel transport to define an *intrinsic notion of the curvature*.

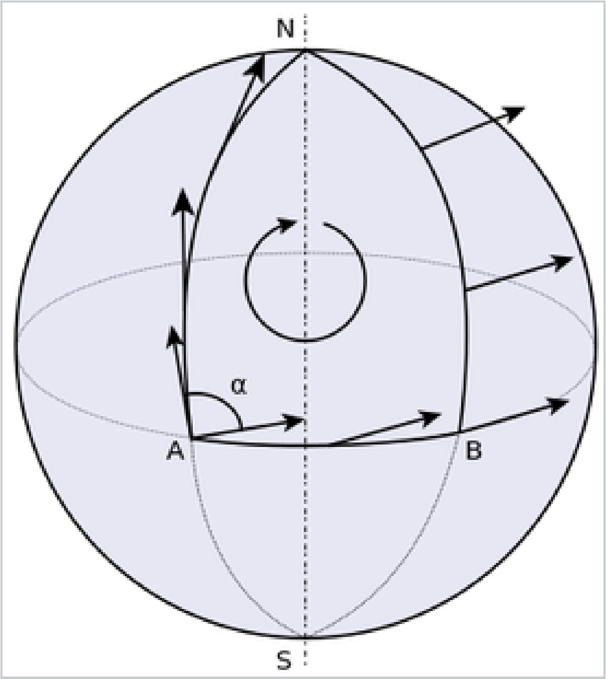
Definition

Let ∇_a be a derivative operator. Let ω_a be a dual vector field and f be a smooth function. We calculate the action of two derivative operators applied to $f\omega_a$:

$$\nabla_a \nabla_b (f\omega_c) = \nabla_a(\omega_c \nabla_b f + f \nabla_a \omega_c) =$$
$$= (\nabla_a \nabla_b f)\omega_c + \nabla_b f \nabla_a \omega_c + \nabla_a f \nabla_b \omega_c + f \nabla_a \nabla_b \omega_c$$

If we substract from this the tensor $\nabla_a \nabla_b (f\omega_c)$, the first three terms of the right-hand side of the definition will cancel the corresponding terms of the expression for $\nabla_a \nabla_b (f\omega_c)$ and we obtain the simple result:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)(f\omega_c) = f(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c.$$



Parallel transport of a vector around a closed loop (from A to N to B and back to A) on the sphere. The angle by which it twists, α, is proportional to the area inside the loop.

- The tensor $(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c$ at point p ***depends only on*** the value of ω_c at p.

→ Consequently, $(\nabla_a \nabla_b - \nabla_b \nabla_a)$ defines a linear map from dual vectors at p to type (0,3) tensors at p; i.e., its action is that of a tensor of type (1,3).

Definition

There exists a a tensor field $R_{abc}{}^{d}$ such that for all dual vector fiels ω_c, we have:

$$\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = R_{abc}{}^{d} \omega_d$$

where $R_{abc}{}^{d}$ is called the *Riemann curvature tensor*.

If we substract from this the tensor $\nabla_a \nabla_b (f \omega_c)$, the first three terms of the right-hand side of the definition will cancel the corresponding terms of the expression for $\nabla_a \nabla_b (f \omega_c)$ and we obtain the simple result:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)(f \omega_c) = f(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c.$$

GEOMETRIC INTERPRETATION OF THE RIEMANN TENSOR

Parallel transport around a small closed curve →

> *„$R_{abc}{}^{d}$ is directly related to the failure of a vector to return to its initial value when parallel transported around a small closed curve.”*

1. We can construct a small closed loop at $p \in M$ by choosing a two-dimensional surface S through p and choosing coordinates t and s in the surface.
2. Let’s consider the loop formed by moving Δt along $s = 0$ curve, folowed by moving Δs along the $\mathrm{t} = \Delta t$ curve, and then moving back by Δt and Δs. (*See bellow.*)
3. Let v^a be a vector at p and let us parallel transport v^a around this closed loop.

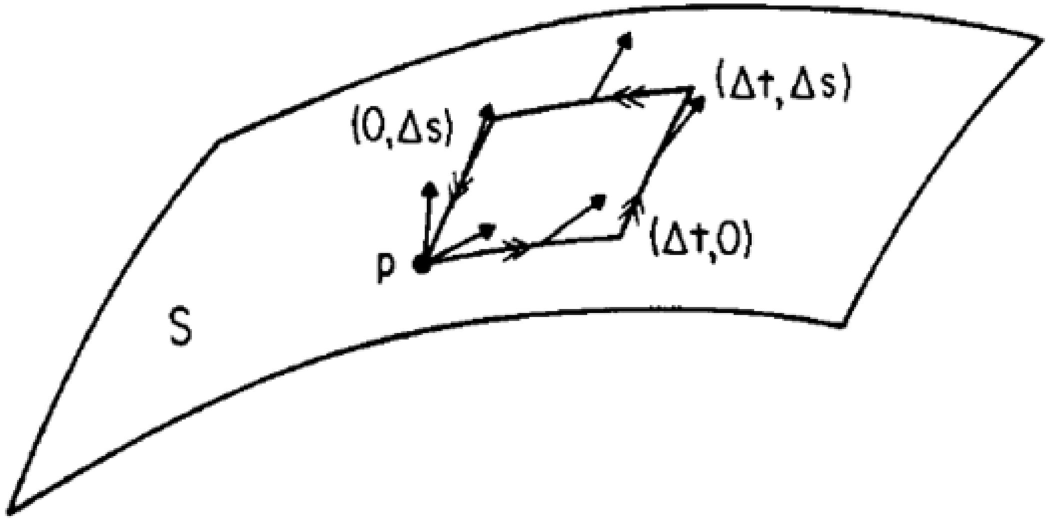


Fig. 3.3. The parallel transport of a vector v^a around a small closed loop. As derived in the text, to second order in Δt and Δs, the change in v^a is governed by the Riemann tensor at p.

Parallel transport (first-order change in $v^a\omega_a$)

It is easiest to ***compute the change*** in v^a when we return to p by letting ω_a be an arbitrary dual vector field and finding the change in the scalar $v^a\omega_a$ as we traverse the loop.

- For small Δt the change, δ_1, in $v^a\omega_a$ on the first leg of the path is

$$\delta_1 = \Delta t \frac{\partial}{\partial t}(v^a\omega_a)\Big|_{(\Delta t/2,0)}$$

- We may ***rewrite the change*** as

$$\delta_1 = \Delta t T^a \nabla_b (v^a\omega_a)\Big|_{(\Delta t/2,0)} = \Delta t v^a T^b \nabla_b \omega_a\Big|_{(\Delta t/2,0)}$$

where T^b is the tangent to the curves of constants s and $T^b\nabla_b v^a = 0$.

Similar expressions hold for the changes δ_2, δ_3 and δ_4 on the other parts of the path. The two „ Δt variations", δ_1 and δ_3, ***combine*** to yield

$$\delta_1 = \Delta t \left(v^a T^b \nabla_b \omega_a\Big|_{(\Delta t/2,0)} - v^a T^b \nabla_b \omega_a\Big|_{(\Delta t/2,\Delta s)} \right),$$

and δ_2 and δ_4 combine similarly.

- Since the term in brackets vanishes as $\Delta s \to 0$, this shows that to first order in Δt and Δs, the total change in $v^a\omega_a$ (and thus the total change in v^a) vanishes.

$\longrightarrow$ ***Parallel transport is path-independent*** to *first-order* in Δt and Δs.

Parallel transport (second-order change in $v^a\omega_a$)

To calculate the second order change in $v^a\omega_a$, we need to evaluate the term in brakets to first order.

Procedure

1. We consider the curve $t = \Delta t/2$ and imagine parallel transporting v^a and $T^b\nabla_b\omega_a$ along this curve from $(\Delta t/2, 0)$ to $(\Delta t/2, \Delta s)$.
2. Now to first order in Δs, v^a at $(\Delta t/2, \Delta s)$ equals the parallel transport of v^a at $(\Delta t/2, 0)$ along this curve, since as remarked above, parallel transport is path-independent to first order.
3. On the other hand, to first order, the term $T^b\nabla_b\omega_a$ at $(\Delta t/2, \Delta s)$ will differ from the parallel transport of that quantity from $(\Delta t/2, 0)$ by the amount $\Delta s\, S^c\nabla_c(T^b\nabla_b\omega_a)$, where S^c is the tangent to the curves of constant t.

→ Thus, to second order in $\Delta t, \Delta s$, we find

$$\delta_1 + \delta_3 = -\Delta t\, \Delta s\, v^a S^c\nabla_c(T^b\nabla_b\omega_a),$$

where, to this accuracy, we may evaluate all tensors at p.

Adding the similar contributions for δ_2 and δ_4, we find the total change in $v^a\omega_a$ is

$$\begin{aligned}\delta(v^a\omega_a) &= \Delta t\ \Delta s\ v^a[T^c\nabla_c(S^b\nabla_b\omega_a) - S^c\nabla_c(T^b\nabla_b\omega_a)] \\ &= \Delta t\ \Delta s\ v^a T^c S^c(\nabla_c\nabla_b - \nabla_b\nabla_c)\omega_a \\ &= \Delta t\ \Delta s\ T^c S^b R_{cba}{}^d\omega_d;\end{aligned}$$

Here we used the fact that the coordinate vector fields T^a and S^a commute

Definition of the Riemann tensor (recall the 3rd slide)

- The above equation can hold for all ω_a if and only if the total change in v^a is

$$\delta v^a = \Delta t\ \Delta s\ v^d\ T^c S^b R_{cbd}{}^a.$$

Accurate to 2nd order in Δt and Δs

The result shows that the Riemann tensor indeed directly measures the path dependence of parallel transport!

We may express the action of the commutator of derivative operators on arbitrary tensor fields in terms of the Riemann tensor. We let ω_a be a dual vector field to find the expression for a vector field t^a.

$$\begin{aligned}0 &= (\nabla_a\nabla_b - \nabla_b\nabla_a)(t^c\omega_c) \\ &= \nabla_a(\omega_c\nabla_b t^c + t^c\nabla_b\omega_c) - \nabla_b(\omega_c\nabla_a t^c + t^c\nabla_a\omega_c) \\ &= \omega_c(\nabla_a\nabla_b - \nabla_b\nabla_a)t^c + t^c(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c \\ &= \omega_c(\nabla_a\nabla_b - \nabla_b\nabla_a)t^c + t^c\omega_d R_{abc}{}^d \quad .\end{aligned}$$

Thus we obtain:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)t^c = -R_{abd}{}^c t^d.$$

PROPERTIES OF RIEMANN TENSOR

By induction, for an arbitrary tensor field ${T^{c_1 \dots c_k}}_{d_1 \dots d_l}$ we find

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) {T^{c_1 \dots c_k}}_{d_1 \dots d_l} = -\sum_{i=1}^{k} {R_{abe}}^{c_i} {T^{c_1 \dots e \dots c_k}}_{d_1 \dots d_l} + \sum_{j=1}^{l} {R_{abd_j}}^{e} {T^{c_1 \dots c_k}}_{d_1 \dots e \dots d_l}$$

Next we establish ***four key properties*** of the Riemann tensor:

1. ${R_{abc}}^d = -{R_{bac}}^d$.
2. ${R_{[abc]}}^d = 0$.
3. For the derivative operator ∇_a naturally associated with the metric, $\nabla_a g_{bc} = 0$, we have
$$R_{abcd} = -R_{abdc}$$
4. The ***Bianchi identity*** holds:
$$\nabla_{[a} {R_{bc]d}}^e = 0.$$

Common forms: ${R^a}_{bcd} + {R^a}_{cdb} + {R^a}_{dbc} = 0$ *(1st Bianchi identity)*
${R^{ab}}_{cd;e} + {R^{ab}}_{de;c} + {R^{ab}}_{ec;d} = 0$ *(2nd Bianchi identity)*

Property (3) follows from *this equation* applied to the metric g_{ab}. → The we find that

$$0 = (\nabla_a \nabla_b - \nabla_b \nabla_a) g_{cd} = {R_{abc}}^e g_{ed} + {R_{abd}}^e g_{ce} = R_{abcd} + R_{abdc}$$

which yields property (3).

It follows from properties (1 – 3) that the Riemann tensor also satisfies the following ***symmetry property***:

$$R_{abcd} = R_{cdab}.$$

Finally, to prove the property (4), i.e. ***Bianchi identity*** , we apply the commutator of derivative operators to the derivative of a dual field vector:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\nabla_c \omega_d = R_{abc}{}^e \nabla_e \omega_d + R_{abd}{}^f \nabla_c \omega_f.$$

On the other hand, we have

$$\nabla_a(\nabla_b \nabla_c \omega_d - \nabla_c \nabla_b \omega_d) = \nabla_a(R_{bcd}{}^e \omega_e) = \omega_e \nabla_a R_{bcd}{}^e + R_{bcd}{}^e \nabla_a \omega_e.$$

If we antisymmetrize over *a*, *b*, and *c* in the last two equations, the left-hand side becomes equal. Equality of the right-hand side yields

$$R_{[abc]}{}^e \nabla_e \omega_d + R_{[ab|d|}{}^f \nabla_{c]} \omega_f = \omega_e \nabla_{[a} R_{bc]d}{}^e + R_{[bc|d|}{}^e \nabla_{a]} \omega_e,$$

The vertical bars indicate that we do not antisymmetrize over d.

The *first term on the left-hand side* vanishes by the equation $R_{[abc]}{}^d = 0$, i.e. property (2), while the second terms on both sides cancel each other. Thus, we obtain for all ω_e,

$$\omega_e \nabla_{[a} R_{bc]d}{}^e = 0,$$

which yields property (4).

DECOMPOSITION OF RIEMANN TENSOR

Ricci tensor and Ricci scalar

It is useful to decompose the Riemann tensor into a ***trace part*** and a ***trace-free part***. By the antisymmetry properties (1) and (3), the trace of the Riemann tensor over its first two or last indices vanishes. However, its trace over the second and fourth indeces defines, the *Ricci tensor*:

$$\boxed{R_{ac} = R_{abc}{}^{b}}$$

R_{ac} satisfies the ***symmetry property***

$$R_{ac} = R_{ca}.$$

The *scalar curvature* is defined as the ***trace*** of the Ricci tensor:

$$R = R_a{}^a.$$

Weyl tensor

The ***trace-free part*** is called the *Weyl tensor*, C_{abcd}, and it is defined for manifolds of dimension $n \geq 3$ by the equation

$$R_{abcd} = C_{abcd} + \frac{2}{n-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) - \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b}.$$

R_{ac} satisfies the ***symmetry property***

$$R_{ac} = R_{ca}.$$

- The Weyl tensor satisfies the symmetric properties (1), (2), and (3) of the Riemann tensor as well as being trace free on all its indices.
- It also bhaves in a very simple manner under conformal transformations of the metric: for this reason is sometimes called the *conformal tensor*.

Contracted Bianchi identity

The contraction of the Bianchi identity leads to an important equation satisfied by R_{ab}:

$$\nabla_a R_{bcd}{}^a + \nabla_b R_{cd} - \nabla_c R_{bd} = 0. \text{ (\textit{Contracted Bianci identity})}$$

Raising the index *d* with the metric and contracting over *b* and *d*, we obtain ***symmetry property***

$$\nabla_a R_c{}^a + \nabla_b R_c{}^b + \nabla_c R = 0$$

or

$$\nabla^a G_{ab} = 0,$$

← *vacuum* if it's equal to 0.
matter if it's not 0!

where

Einstein tensor → $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}.$

- The Einstein tensor (and the twice contracted Bianchi identity) plays an important role in the *Einstein equation* which relate the ***geometry of spacetime*** to the ***distribution of matter*** within it. (As we shall see later!)

Thank you for your attention!

Lecture notes will be available shortly at https://wigner.hu/~barta/GRcourse2020/

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