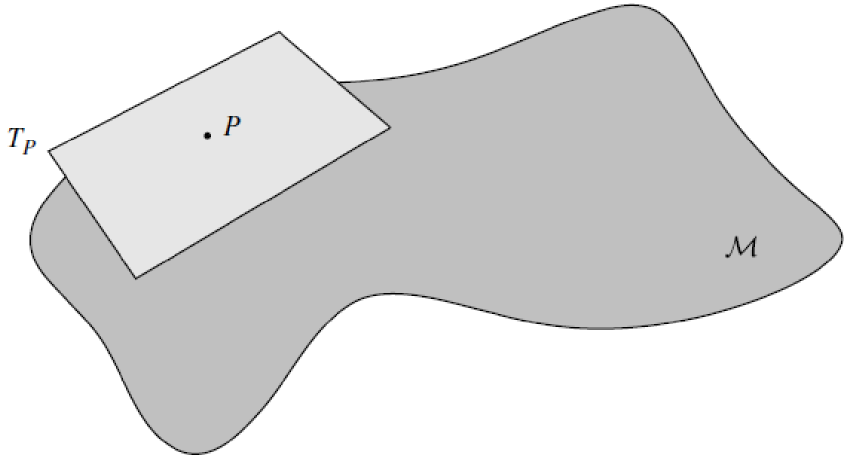
INTRODUCTION TO GENERAL RELATIVITY

Tangent space, dual space

WHAT IS A TANGENT SPACE?



The tangent plane T_P to the curved surface M at the point P.

Tangent spaces to manifolds:

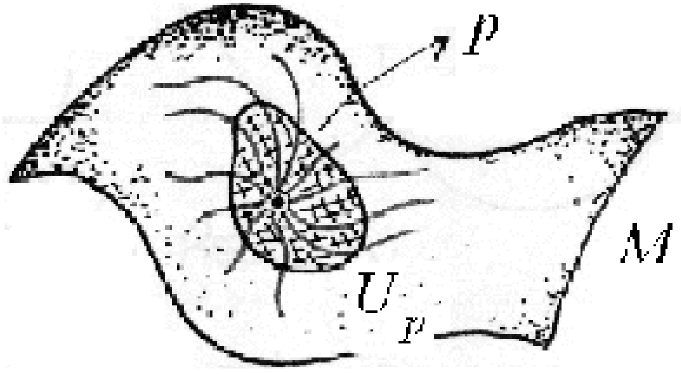
To aid our intuition of local Cartesian coordinates, it is useful to consider the simple example of a two-dimensional Riemannian manifold, which we can often consider as a generally curved surface embedded in three-dimensional Euclidean space.

A simple example is the surface of a sphere, shown in the left-hand side Figure.

Informally:

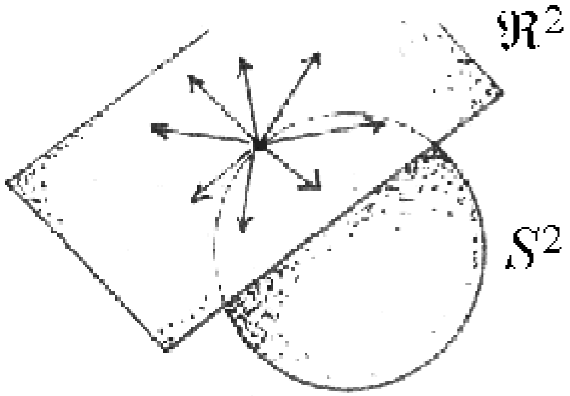
At any arbitrary point P we can find coordinates x and y (say) such that in the neighbourhood of P we have $ds^2 = dx^2 + dx^2$. It thus follows that a Euclidean two-dimensional space (a plane) will match the manifold locally at P. This Euclidean space is called the *tangent space* T_P to the manifold at P.

TANGENT SPACES AND VECTOR FIELDS



In a neighborhood U_p of each point $p \in \mathcal{M}$

we consider the curves that go through p



The tangent space in a generic point of an $\mathbb{S}^2$ sphere

$$\boxed{\forall p \in \mathcal{M} : p \mapsto T_p\mathcal{M} \quad \dim T_p\mathcal{M} = m}$$

$T_p\mathcal{M}$ parametrizes the possible directions in which a curve starting at p can depart.

$$\vec{t}_p = c^\mu \frac{\vec{\partial}}{\partial x^\mu} \ni T_p\mathcal{M}$$

A tangent vector is a 1st order differential operator

Under change of local coordinates

$$x^\mu = x^\mu(y) \qquad ; \qquad y^\alpha = y^\alpha(x)$$

$$\vec{t}_p = c^\mu \left(\frac{\partial y^\alpha}{\partial x^\mu}\right) \frac{\vec{\partial}}{\partial y^\alpha} = c^\alpha \frac{\vec{\partial}}{\partial y^\alpha}$$

$$c^\alpha \equiv c^\mu \left(\frac{\partial y^\alpha}{\partial x^\mu}\right)$$

Theorem

Let M be an n-dimensional manifold. Let $p \in \mathcal{M}$ and let T_p denote the tangent space at p. Then $\dim T_p = n$.

Proof: We shall show that $\dim T_p = n$ by constructing a basis of T_p i.e., by finding n linearly independent tangent vectors which span T_p. Let $\psi: O \to U \in \mathbb{R}^n$ be a chart with $p \in O$ (see Fig. bellow). If $f \in \mathcal{F}$, then by definition $f \circ \psi^{-1}: U \to \mathbb{R}$ is C^∞. For $\mu = 1, \dots, n$ define $X_\mu: \mathcal{F} \to \mathbb{R}$ by

$$X_\mu(f) = \frac{\partial}{\partial x^\mu}(f \circ \psi^{-1})\bigg|_{\psi(p)}$$

where $(x^1, \dots, x^n)$ are the Cartesian coordinates of $\mathbb{R}^n$. Then $(X^1, \dots, X^n)$ are tangent vectors, and it is easily sen that they are linearly independent. (To show that they span T_p – along with the rest of the proof – see Wald p. 16.)

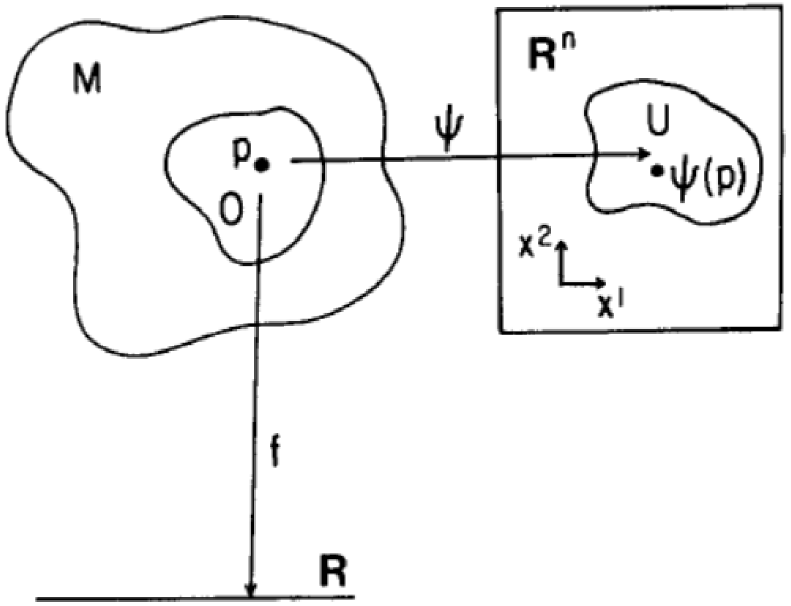


Illustration for the directional derivates, X_μ used in the theorem.

Definition

The basis $\{X_\mu\}$ of T_p is called a *coordinate basis* where X_μ is commonly denoted by $\partial/\partial x^\mu$. If one choses a different chart, ψ', an other coordinate basis $\{X'_\nu\}$ is obtained.

Definition

One can express X_μ in terms of the new basis $\{X'_\nu\}$ by using the chain rule:

$$X_\mu = \sum_{\nu=0}^{n} \left.\frac{\partial x'^\nu}{\partial x^\mu}\right|_{\psi(p)} X'_\nu \ ,$$

where x'^ν denotes the νth component of the map $\psi' \circ \psi^{-1}$. Consequently, the components of v'^ν of a vector v in the new coordinate basis are related to the components v^μ in the old basis by

$$v'^\nu = \sum_{\mu=0}^{n} v^\mu \frac{\partial x'^\nu}{\partial x^\mu} \ .$$

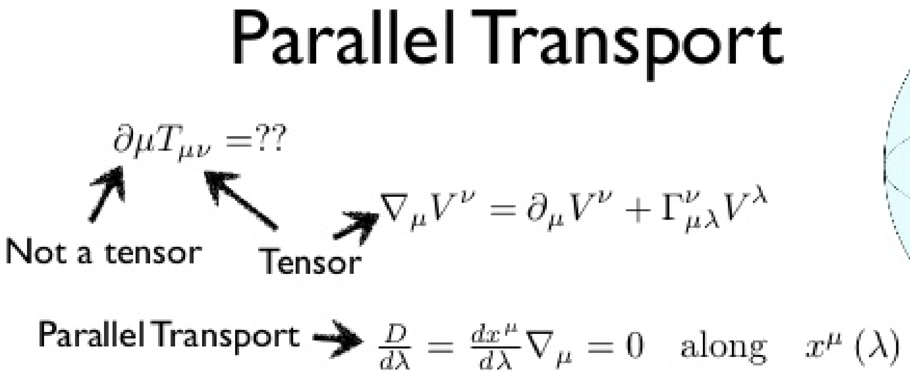
The above equation is known as the *vector transformation law*.

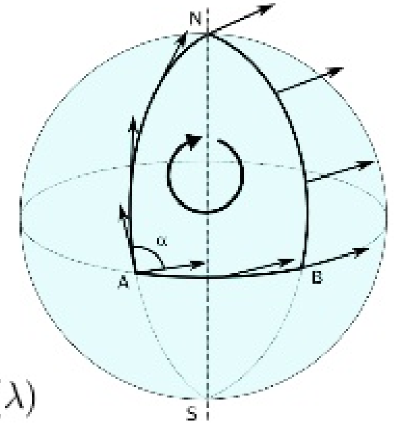
Generalization: In any coordinate basis, the components of the tangent vector $T^\mu \in T_p$ to a *smooth curve* ($\mathcal{C}^\infty$) are given by

$$T^\mu = \frac{dx^\mu}{dt} \ .$$

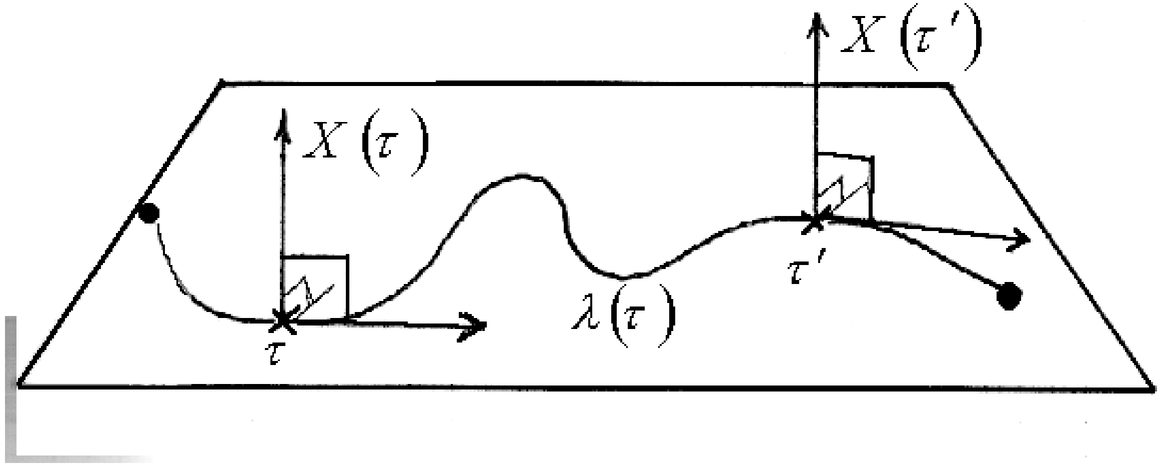
REMARKS

- In the discussion above, we fixed a point $p \in \mathcal{M}$ and considered the tangent space, T_p at p. At another point $q \in \mathcal{M}$ one may as well define T_q.
- It is important to emphasize that, given only the structure of a manifold, there is no natural way of identifying T_q with T_p; that is, there is no way of determining whether a tangent vector at q is the "same" as a tangent vector at p.
- After the next lecture, we will see that when additional structure (namely, a connection or derivate operator) is given on the manifold, one has a notation "*parallel transport*" of vectors from p to q along a curve joining these points. $\rightarrow$ *If the curvature is non-zero*, the, the identification of T_q with T_p will depend on the choise of curve.

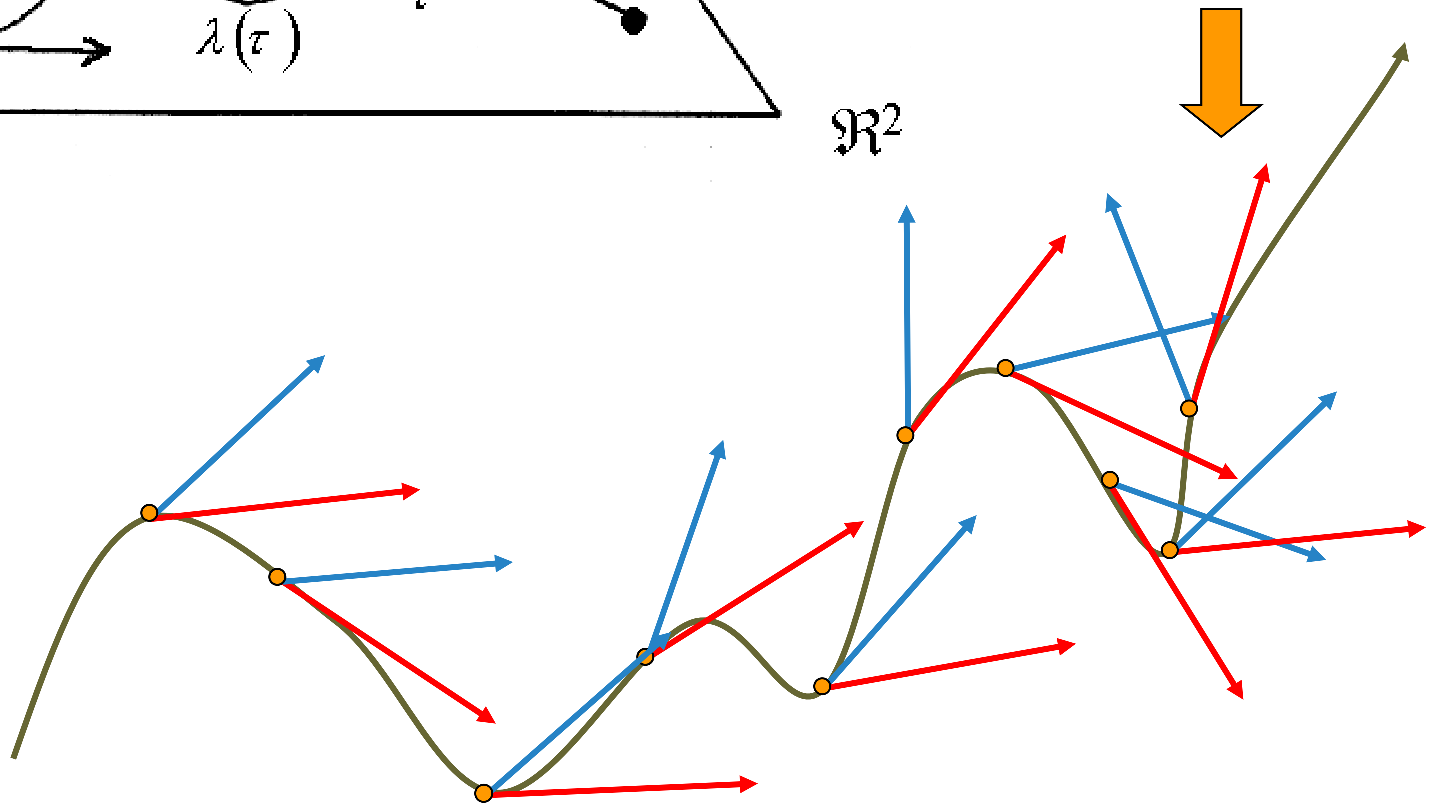




PARALLEL TRANSPORT



A vector field is parallel transported along a curve, when it mantains a constant angle with the tangent vector to the curve



Definition

A *tangent field* v on a manifold $\mathcal{M}$ is an assignment of a tangent vector $v|_p \in T_p$, at each point $p \in \mathcal{M}$. Despite the fact that the tangent spaces T_p and T_q at different points are different vector spaces, there is a natural notation of what it means for v to vary smoothly from point to point. If f is a smooth ($\mathcal{C}^\infty$) function , then each $p \in \mathcal{M}, v|_p \in T_{p(f)}$ is a number, i.e., $v(f)$ is a function on $\mathcal{M}$.

Theorem

The tangent field v is said to be *smooth* if for each function f, the function $v(f)$ is also smooth. Since the coordinate basis field X_μ are easily verified to be smooth $\longrightarrow$ a vector field v is smooth if and only if its coordinate basis components, v^μ, are smooth functions.

We described the tangent vectors as "infinitesimal displacements". Let's see the precise meaning given to this picture!

Definition

A *one-parameter group of diffeomorphisms* ϕ_t is a C^∞ map from $\mathbb{R} \times \mathcal{M} \to \mathcal{M}$ such that for fixed $t \in \mathbb{R}$, $\phi_t: \mathcal{M} \to \mathcal{M}$ is a diffeomorphism and for all $t, s \in \mathbb{R}$, we have $\phi_t \circ \phi_s = \phi_{t+s}$.

(The relation implies that $\phi_{t=0}$ is the identity map.)

- We can associate a vector field v to the diffeomorphism ϕ_t as follows: For fixed $p \in \mathcal{M}$, $\phi_t(p): \mathbb{R} \to \mathcal{M}$ is a curve, called the *orbit* of ϕ_t, which passes through p at $t = 0$.
- Let's define $v|_p \in T_{p(f)}$ to be the tangent to this curve at $t = 0$. Then v, associated to a one-parameter group of (finite) transformations of $\mathcal{M}$, can be thought of as the infinitesimal generator of these transformations.

Definition

A family of curves in $\mathcal{M}$ is called *integral curves* if one and only one curve passes through each point $p \in \mathcal{M}$ and the tangent to this curve is $v|_p$ at p.

- If we pick a coordinate system in the neighbourhood of p, we see that the problem of finding such curves reduces to solving the system,

$$\frac{dx^\mu}{dt} = v^\mu(x^1, \dots, x^n)$$

of ordinary differential equations in $\mathbb{R}^n$, where v^μ is the μth component of v in the coordinate basis $\{\partial/\partial x^\mu\}$.

Such a system of equations has a unique solution given a starting point at $t = 0$, and thus every smooth vector field v has a unique family of integral curves.

DUAL SPACES AND ONE-FORMS

Previously we defined vector as a basis-dependent set of numbers which transform according to above learned ways when a different basis is chosen. ← *Although, this description is mathematically correct but it hides the geometric nature of vectors!*

General Relativity indeed operates with geometric (i.e., basis-independent) objects. *Dual spaces* and *dual vectors* sneak into GR and, therefore, let's have a closer look at them.

Definition

Given a vector space $\boldsymbol{V}$, we define its dual space $\boldsymbol{V}^*$ to be the set of all linear transformations $\phi: \boldsymbol{V} \rightarrow \boldsymbol{V}^*$. The ϕ is called a *linear functional*.

- In other words, φ maps a vector $\vec{v} \in \boldsymbol{V}$ to and element of $\mathbb{F}$ (lets just assume that $\mathbb{F} = \mathbb{R}$). If you take all the possible ways that a φ can take in such vectors and produce real numbers, we get $\boldsymbol{V}^*$.

Definition

A *one-form* is a linear function f on a linear space if

$$f(\alpha\vec{a} + \beta\vec{b}) = \alpha f(\vec{a}) + \beta f(\vec{b})$$

for any real numbers α, β and vectors $\vec{a}$ and $\vec{b}$.

EXAMPLES OF DUAL SPACES

Here is a list of examples of dual spaces:

- Example 1: Let $V = \mathbb{R}^3$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$, then $\varphi(x, y, z) = 2x + 3y + 4z$ is a member of V^*.
- Example 2: Let $V = P_n$ (the set of polynomials with degreee n) and $\varphi : P_n \to \mathbb{R}$, then $\varphi(p) = p(1)$ is a member of V^*. Concretely, $\varphi(1 + 2x + 3x^2) = 1 + 2 \cdot 1 + 3 \cdot 1^2 = 6$.
- Example 3: Let $V = M_{m \times n}$ (the set of matrices with dimensions $m \times n$) and $\varphi : M_{m \times n} \to \mathbb{R}$, then $\varphi(A) = \mathrm{Trace}(A)$ is a member of V^*. In specific,

$$\varphi\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right) = 1 + 5 = 6$$

- Example 4: Let $V = C([0, 1])$ (the set of all continuous function on the interval $[0, 1]$) and $\varphi : C[(0, 1)] \to \mathbb{R}$, then $\varphi(g) = \int_0^1 g(x)\mathrm{d}x$ is a member of V^*. For instance, $\varphi(e^x) = \int_0^1 e^x \mathrm{d}x = e^1 - 1 = e - 1$.

As it turns out, the elements of V^* satisfy the axioms of a vector space and therefore V^* is indeed a vector space itself.

DUAL BASIS

If $b = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ is a basis of vector space V, then $b^* = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is a basis of V^*. If you define φ via the following relations, then the basis you get is called the *dual basis*:

$$\varphi_i(\underbrace{a_1\mathbf{v_1} + \cdots + a_n\mathbf{v_n}}_{\text{A vector } \mathbf{v}\in V, a_i\in\mathbb{F}}) = a_i, \quad i = 1, \dots, n$$

It is as if the functional φ_i acts on a vector $\mathbf{v} \in V$ and returns the i-th component a_i. Another way to write the above relations is if you set $\varphi_i(\mathbf{v_j}) = \delta_{ij}$.

Then any functional φ can be written as a linear combination of the dual basis vectors, i.e.

$$\boxed{\varphi = \varphi(\mathbf{v_1})\varphi_1 + \varphi(\mathbf{v_2})\varphi_2 + \ldots + \varphi(\mathbf{v_n})\varphi_n}$$

EXAMPLES OF DUAL BASIS

Let's see a concrete example. Assume $\boldsymbol{V} = \mathbb{R}^2$ and a vector basis $\boldsymbol{b} = \{(2,1), (3,1)\}$, then what is the dual basis $\boldsymbol{b}^*$?

By definition, it's $\varphi_i(\mathbf{v_j}) = \delta_{ij}$, therefore:

$$\varphi_1(\mathbf{v_1}) = \delta_{11} = 1 \Leftrightarrow \varphi_1(2,1) = 1 \Leftrightarrow \varphi_1\left[2(1,0) + 1(0,1)\right] = 1 \Leftrightarrow 2\varphi_1(1,0) + 1\varphi_1(0,1) = 1$$
$$\varphi_1(\mathbf{v_2}) = \delta_{12} = 0 \Leftrightarrow \varphi_1(3,1) = 0 \Leftrightarrow \varphi_1\left[3(1,0) + 1(0,1)\right] = 0 \Leftrightarrow 3\varphi_1(1,0) + 1\varphi_1(0,1) = 0$$

If you solve the system:

$$\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \varphi_1(1,0) \\ \varphi_1(0,1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

You get $\varphi_1(1,0) = -1, \quad \varphi_1(0,1) = 3$. Therefore:

$$\varphi_1(x,y) = x\varphi_1(1,0) + y\varphi_1(0,1) = -x + 3y$$

Similarly one can prove that:

$$\varphi_2(x,y) = x\varphi_2(1,0) + y\varphi_2(0,1) = x - 2y$$

Therefore the dual basis b^* is equal to $\{\varphi_1, \varphi_2\} = \{-x + 3y, x - 2y\}$. Now here comes the magic. Suppose that you have a function $\varphi = 8x - 7y$ and you would like to write it as a linear combination of the dual basis. How would you do?

$$\varphi = \varphi(\mathbf{v_1})\varphi_1 + \varphi(\mathbf{v_2})\varphi_2 + \ldots + \varphi(\mathbf{v_n})\varphi_n$$

Where $\mathbf{v_1} = (2, 1)$ and $\mathbf{v_2} = (3, 1)$. Let us do the math:

$$\begin{aligned}
8x - 7y &= \overbrace{\varphi(2,1)}^{\varphi(\mathbf{v_1})} \cdot \underbrace{(-x + 3y)}_{\varphi_1} + \overbrace{\varphi(3,1)}^{\varphi(\mathbf{v_2})} \cdot \underbrace{(x - 2y)}_{\varphi_2} \\
8x - 7y &= (8 \cdot 2 - 7 \cdot 1) \cdot (-x + 3y) + (8 \cdot 3 - 7 \cdot 1) \cdot (x - 2y) \\
8x - 7y &= 9(-x + 3y) + 17(x - 2y) \\
8x - 7y &= 8x - 7y
\end{aligned}$$

THE DUAL OF A DUAL SPACE

Recall that the dual of space is a vector space on its own right, since the linear functionals φ satisfy the axioms of a vector space. But if V^* is a vector space, then it is perfectly legitimate to think of its dual space, just like we do with any other vector space. This might feel too recursive, but hold on. The double dual space is $(V^*)^* = V^{**}$ and is the set of all *linear transformations* $\varphi : V^* \to \mathbb{F}$.

ISOMORPHISMS

When we defined V^* from V we did so by picking a special basis (the dual basis), therefore the isomorphism from V to V^* is not canonical. It turns out that the isomorphism between the initial vetor space V and its double dual, V^{**}, is canonical as we shall see right away. Let $v \in V, \varphi \in V^*$ and $\hat{v} \in V^{**}$. We can now define a linear map:

$$\hat{u}(\varphi) = \varphi(u)$$

This is a canonical isomorphism between V and V^{**}. Mind that we are talking about finite dimensional vector spaces V, i.e. $\dim(V) < \infty$.

OUTLOOK: CONNECTION TO GENERAL RELATIVITY

Consider the space of partial derivative operators. How do we know that this is a vector space? Let's check:

$$\frac{\partial}{\partial x^k}(af+bg) = \frac{\partial(af+bg)}{\partial x^k} = a\frac{\partial f}{\partial x^k} + b\frac{\partial g}{\partial x^k} = a\frac{\partial}{\partial x^k}(f) + b\frac{\partial}{\partial x^k}(g)$$

Then let us pick up a basis:

$$b = \left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}\right\}$$

And then let us define its dual space $b^* = \{\mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^n\}$. By definition the functionals $\mathrm{d}x^i$ must fulfill the following relations:

$$\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$$

So, $\mathrm{d}x$'s in reality are linear functionals that act on elements of the partial derivatives vector space. They are not be thought as scalars, i.e. as infinitesimal displacements along the coordinate axis.

So, $\mathbf{d}x$'s in reality are linear functionals that act on elements of the partial derivatives vector space. They are not be thought as scalars, i.e. as infinitesimal displacements along the coordinate axis.

Now suppose we have a scalar function $f(x)$ and we define its total differential as:

$$\mathrm{d}f = \frac{\partial f}{\partial x^1}\mathrm{d}x^1 + \frac{\partial f}{\partial x^2}\mathrm{d}x^2 + \ldots + \frac{\partial f}{\partial x^n}\mathrm{d}x^n$$

And we would like to calculate the directional derivative of $f(x)$ along the direction of some vector $v \in V$, that is calculate the rate of change of $f(x)$ along v.

$$\mathrm{d}f(v) = \frac{\partial f}{\partial x^i}\mathrm{d}x^i\left(v^j \frac{\partial}{\partial x^j}\right)$$

Recall now that the basis vectors $\mathbf{d}x$'s were chosen in such a way that when act upon $\frac{\partial}{\partial x^j}$ they "select" the i-th component. Therefore:

$$\mathrm{d}f(u) = \frac{\partial f}{\partial x^i}v^i$$

BIBLIOGRAPHY

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Now, that we have succesfully survived today’s lecture…

Next week: let’s resume with *tensors*

Lecture notes on the course and other materials (e.g. recommended text books) will be available shortly at
https://wigner.hu/~barta/GRcourse2020/

Dániel Barta
E-mail: barta.daniel@wigner.hu